

# Semiclassical wave-packets emerging from interaction with an environment

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## Abstract

We study the quantum evolution in dimension three of a system composed by a test particle interacting with an environment made of  $N$  harmonic oscillators. At time zero the test particle is described by a spherical wave, i.e. a highly correlated continuous superposition of states with well localized position and momentum, and the oscillators are in the ground state. Under suitable assumptions on the physical parameters characterizing the model, we give an asymptotic expression of the solution of the Schrödinger equation of the system with an explicit control of the error. The result shows that the approximate expression of the wave function is the sum of two terms, orthogonal in  $L^2(\mathbb{R}^{3(N+1)})$  and describing rather different situations.

In the first one all the oscillators remain in their ground state and the test particle is described by the free evolution of a slightly deformed spherical wave.

The second one consists of a sum of  $N$  terms where in each term there is only one excited oscillator and the test particle is correspondingly described by the free evolution of a wave packet, well concentrated in position and momentum. Moreover the wave packet emerges from the excited oscillator with an average momentum parallel to the line joining the oscillator with the center of the initial spherical wave. Such wave packet represents a semiclassical state for the test particle, propagating along the corresponding classical trajectory.

The main result of our analysis is to show how such a semiclassical state can be produced, starting from the original spherical wave, as a result of the interaction with the environment.

## 1. INTRODUCTION

The analysis of the emergence of a classical behavior in a quantum system is a subject of interest not only from the conceptual but also from the applicative point of view. Indeed, for the correct behavior of any quantum device it is surely relevant to understand under which conditions genuine quantum effects, like interference or entanglement, are reduced or cancelled. It is widely accepted that a crucial classicality condition is obtained requiring that the Planck's constant  $\hbar$  is small with respect to the typical action of the system. An important example where the classical behavior is recovered in the limit  $\hbar \rightarrow 0$  is the case of the evolution of a quantum particle initially described by a "semiclassical" state, e.g. a state with a suitable dependence on  $\hbar$  like a WKB or a coherent state. A typical result in this context is that for  $\hbar \rightarrow 0$  the evolution is again given by a semiclassical state, propagating along the corresponding classical trajectories (see e.g. [R], [CR]).

On the other hand, it is important to remark that in many interesting physical situations the initial state of the quantum system is a genuine quantum state, like a superposition state, and nevertheless the system exhibits a classical behavior during the evolution. In such cases the limit  $\hbar \rightarrow 0$  for the isolated system cannot help and the role of the quantum environment must be taken into account, according to the approach known as decoherence theory (see e.g. [GJKKSZ], [BGJKS], [H]; for some rigorous results see [AFFT], [CCF]). Roughly speaking, for such kind of physical situations one has to consider the Schrödinger equation for the "system + environment" and one has to prove that, for appropriate values of the physical parameters characterizing system and environment, the system shows a classical behavior as a result of its interaction with the environment.

In this paper we consider the case of the evolution of a quantum particle initially described by a spherical wave, i.e. a highly correlated continuous superposition of states with well localized position and momentum. Our aim is to show that a semiclassical wave packet for the particle, and therefore the propagation along a classical path, emerges as a result of the interaction with the environment.

This kind of problem was already raised in the early days of Quantum Mechanics and it was discussed in a seminal paper by Mott in 1929 ([M]) in connection with a possible explanation of the linear tracks left by an  $\alpha$ -particle in a cloud chamber (see also [FigT] for a historical analysis of the problem). We recall that the  $\alpha$ -particle is emitted by a radioactive source in the form of a spherical wave and then it interacts with the atoms of the vapor filling the device. If an atom is excited then the amplified effect of the excitation is observed. As a matter of fact, in the experiment one observes straight tracks, corresponding to the excitation of a sequence of many atoms whose positions are aligned with the center of the spherical wave. The observed track is regarded as the experimental manifestation of the "classical trajectory" of the  $\alpha$ -particle. From the point of view of a theoretical description, the non trivial problem arises to explain how a spherical wave can produce the observed classical trajectory. In Mott's paper an answer is given considering a model of environment made of only two atoms. At a physical level of rigor, it is shown that the probability that both atoms are excited is negligible unless the two atoms are aligned with the center of the spherical wave. The approach is based on second order

perturbation theory for the stationary Schrödinger equation and a repeated use of stationary phase arguments (we refer to [DFT] for an attempt to revisit Mott's approach in a fully time dependent setting and to [FinT] for a detailed analysis of a simpler one dimensional model).

Here we consider a more general quantum system in  $\mathbb{R}^3$  made of a test particle, initially described by a spherical wave centered in the origin, interacting with  $N$  harmonic oscillators, initially placed in their ground state. The aim is to study the evolution of the whole system after the interaction of the test particle with each oscillator has taken place. More precisely, we introduce a set of assumptions on the physical parameters characterizing the system, collectively described by a small parameter  $\varepsilon > 0$ , and we give an asymptotic expression for the solution of the Schrödinger equation up to order  $\varepsilon^2$  with an explicit control of the error.

Roughly speaking, the result shows that the probability that more than one oscillator is excited is negligible and therefore the evolution of the system can be decomposed in only two rather different "histories". In the first one the oscillators remain in their ground state and the test particle is described by the free evolution of a (slightly deformed) spherical wave. The second history consists of a sum of  $N$  terms where in each term there is only one excited oscillator. Here the test particle is correspondingly described by the free evolution of a wave packet, well concentrated in position and momentum, emerging from the excited oscillator with an average momentum parallel to the line joining the origin with the oscillator.

Such wave packet represents a semiclassical state for the test particle, propagating along the corresponding classical trajectory (the straight track observed in the cloud chamber).

We stress that the main result of our analysis is to show how such a semiclassical state can be produced, starting from the original spherical wave, as a result of an interaction with the environment. Moreover we emphasize that the interaction with the environment is entirely described in terms of the Schrödinger dynamics without any recourse to wave packet collapse rule.

The paper is organized as follows.

In section 2 we give a detailed description of the model and we formulate our main result (theorem 2.1). In section 3 we describe the line of the proof and we formulate some intermediate results required to conclude the proof of theorem 2.1. In sections 4, 5, 6, 7, 8 we give a proof of the above mentioned intermediate results.

For the convenience of the reader, we collect here some of the most used notation in the paper.

- $\mathbf{x} = (x_1, x_2, x_3)$  is a vector in  $\mathbb{R}^3$ ,  $|\mathbf{x}|$  the euclidean norm and  $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$  the corresponding unit vector. The scalar product in  $\mathbb{R}^3$  is  $\mathbf{x} \cdot \mathbf{y}$ .
- $\underline{n} = (n_1, n_2, n_3)$  is a vector in  $\mathbb{N}^3$  and, with an abuse of notation,  $|n| = n_1 + n_2 + n_3$ .
- $\langle \mathbf{x} \rangle = (1 + |\mathbf{x}|^2)^{1/2}$ ,  $\mathbf{x} \in \mathbb{R}^3$  and  $\langle y \rangle = (1 + y^2)^{1/2}$ ,  $y \in \mathbb{R}$ .
- $\|\cdot\|$ ,  $(\cdot, \cdot)$  are the norm and the scalar product in  $L^2(\mathbb{R}^{3(N+1)})$  respectively.
- $\|\cdot\|_{L^p}$ ,  $p > 0$ , is the norm in  $L^p(\mathbb{R}^3)$  and  $\langle \cdot, \cdot \rangle$  the scalar product in  $L^2(\mathbb{R}^3)$ .
- The derivative of a function  $f$  defined in  $\mathbb{R}^3$  is denoted by

$$D^\alpha f(\mathbf{x}) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} f(\mathbf{x})$$

for  $\underline{\alpha} \in \mathbb{N}^3$  and  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ .

-  $W_s^{n,1}(\mathbb{R}^3)$ ,  $n \in \mathbb{N}$ ,  $s > 0$ , is the weighted Sobolev space equipped with the norm

$$\|f\|_{W_s^{n,1}} = \sum_{\underline{\alpha}, |\alpha| \leq n} \int d\mathbf{x} \langle \mathbf{x} \rangle^s |D^{\underline{\alpha}} f(\mathbf{x})|$$

-  $\tilde{f}$  is the Fourier transform of  $f$ .

- We shall use the following abbreviations for sums and products

$$\sum_{\underline{n}} \equiv \sum_{n_1, n_2, n_3=1}^{\infty} \quad \prod_{k, k \neq j} \equiv \prod_{k=1, k \neq j}^N$$

- During the proofs we shall often denote by  $c$ ,  $c_k$  a generic positive constant, possibly dependent on the integer  $k$ .

## 2. DESCRIPTION OF THE MODEL AND MAIN RESULT

Let us consider a non relativistic quantum system made of  $N+1$  spinless particles in dimension three, where one test particle has mass  $M$  and the remaining  $N$  particles with mass  $m$  are bound by a harmonic potential of frequency  $\omega$  around the equilibrium positions  $\mathbf{a}_1, \dots, \mathbf{a}_N$ . In the model, the test particle plays the role of the  $\alpha$ -particle while the harmonic oscillators play the role of electrons in a simplified version of model-atoms with fixed nuclei. The interaction between the test particle and the  $j$ -th harmonic oscillator is described by a smooth two body potential  $V$ . We denote by  $\mathbf{R}$  the position coordinate of the test particle and by  $\mathbf{r}_1, \dots, \mathbf{r}_N$  the position coordinates of the harmonic oscillators. Therefore the Hamiltonian of the system in  $L^2(\mathbb{R}^{3(N+1)})$  is given by

$$H = H_0 + \lambda \sum_{j=1}^N V_j \tag{2.1}$$

where

$$H_0 = h_0 + \sum_{j=1}^N h_j, \quad h_0 = -\frac{\hbar^2}{2M} \Delta_{\mathbf{R}}, \quad h_j = -\frac{\hbar^2}{2m} \Delta_{\mathbf{r}_j} + \frac{1}{2} m \omega^2 (\mathbf{r}_j - \mathbf{a}_j)^2 \tag{2.2}$$

$\lambda > 0$  is a coupling constant and  $V_j$  is the multiplication operator by

$$V_j(\mathbf{R}, \mathbf{r}_j) = V(\delta^{-1}(\mathbf{R} - \mathbf{r}_j)), \quad \delta > 0 \tag{2.3}$$

We are interested in the evolution of the system when the initial state is given in the product form

$$\Psi_0(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \psi(\mathbf{R}) \prod_{j=1}^N \varphi_{\mathbf{0},j}(\mathbf{r}_j) \tag{2.4}$$

where  $\psi(\mathbf{R})$  is a spherical wave centered in the origin and  $\varphi_{\underline{0},j}(\mathbf{r}_j)$  is the ground state of the harmonic oscillator centered in  $\mathbf{a}_j$ . More precisely the spherical wave is given by

$$\psi(\mathbf{R}) = \mathcal{N} f(\sigma^{-1}\mathbf{R}) \int_{S^2} d\hat{\mathbf{u}} e^{i\frac{Mv_0}{\hbar}\hat{\mathbf{u}}\cdot\mathbf{R}} \quad (2.5)$$

where  $\mathcal{N}$  is a normalization constant,  $\sigma, v_0 > 0$ ,  $f$  belongs to the Schwartz space  $\mathcal{S}(\mathbb{R}^3)$  with  $\|f\|_{L^2} = 1$  and it is rotationally invariant. For the sake of concreteness we choose

$$f(\mathbf{x}) = \pi^{-3/4} e^{-\frac{|\mathbf{x}|^2}{2}} \quad (2.6)$$

but it will be clear in the following that the result of our analysis is independent of the specific choice of  $f$ . Formula (2.5) defines a spherical wave concentrated in position around the origin with an isotropic momentum  $Mv_0$ . Moreover, for the eigenfunctions of the harmonic oscillators, we denote  $\underline{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$  and

$$\varphi_{\underline{n},j}(\mathbf{r}) = \varphi_{\underline{n}}(\mathbf{r} - \mathbf{a}_j) = \gamma^{-3/2} \phi_{\underline{n}}(\gamma^{-1}(\mathbf{r} - \mathbf{a}_j)), \quad \gamma = \sqrt{\frac{\hbar}{m\omega}} \quad (2.7)$$

$$\phi_{\underline{n}}(\mathbf{x}) \equiv \phi_{n_1}(x_1)\phi_{n_2}(x_2)\phi_{n_3}(x_3) \quad (2.8)$$

where  $\phi_{n_k}$  is the Hermite function of order  $n_k$ . In particular the ground state corresponds to  $\underline{n} = \underline{0} = (0, 0, 0)$ .

In this generality, it is surely too hard to obtain a detailed description of the evolution of the system. We shall limit ourselves to consider an appropriate scaling limit and to derive an approximate evolution with an explicit control of the error. More precisely, we introduce a small parameter  $\varepsilon > 0$  and fix

$$\hbar = \varepsilon^2 \quad M = 1 \quad \sigma = \varepsilon \quad m = \varepsilon \quad \omega = \varepsilon^{-1} \quad \delta = \varepsilon \quad \lambda = \varepsilon^2 \quad (2.9)$$

Under this scaling the Hamiltonian becomes

$$H^\varepsilon = H_0^\varepsilon + \varepsilon^2 V^\varepsilon \quad (2.10)$$

where

$$H_0^\varepsilon = h_0^\varepsilon + \sum_{j=1}^N h_j^\varepsilon, \quad h_0^\varepsilon = -\frac{\varepsilon^4}{2} \Delta_{\mathbf{R}}, \quad h_j^\varepsilon = \varepsilon^{-1} \left[ -\frac{\varepsilon^4}{2} \Delta_{\mathbf{r}_j} + \frac{1}{2} (\mathbf{r}_j - \mathbf{a}_j)^2 \right] \quad (2.11)$$

and

$$V^\varepsilon = \sum_{j=1}^N V_j^\varepsilon, \quad V_j^\varepsilon(\mathbf{R}, \mathbf{r}_j) = V(\varepsilon^{-1}(\mathbf{R} - \mathbf{r}_j)) \quad (2.12)$$

The rescaled initial state of the system is

$$\Psi_0^\varepsilon(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \psi^\varepsilon(\mathbf{R}) \prod_{j=1}^N \varphi_{0,j}^\varepsilon(\mathbf{r}_j) \quad (2.13)$$

$$\psi^\varepsilon(\mathbf{R}) = \frac{\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} f(\varepsilon^{-1} \mathbf{R}) \int_{S^2} d\hat{\mathbf{u}} e^{\frac{i}{\varepsilon^2} v_0 \hat{\mathbf{u}} \cdot \mathbf{R}} \quad (2.14)$$

$$\varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) = \frac{1}{\varepsilon^{3/2}} \phi_{\underline{n}}(\varepsilon^{-1}(\mathbf{r}_j - \mathbf{a}_j)) \quad \underline{n} \in \mathbb{N}^3 \quad (2.15)$$

By a direct computation one sees that the normalization constant  $\mathcal{N}_\varepsilon$  for  $\varepsilon \rightarrow 0$  reduces to

$$\mathcal{N}_0 = \frac{v_0}{4\pi} \quad (2.16)$$

We notice that under this scaling the energy levels of each harmonic oscillator are

$$E_{\underline{n}}^\varepsilon = \varepsilon \left( |\underline{n}| + \frac{3}{2} \right), \quad |\underline{n}| = n_1 + n_2 + n_3 \quad (2.17)$$

Furthermore we introduce the following assumptions:

- (A) *The Fourier transform  $\tilde{V}$  of the interaction potential  $V$  belongs to the weighted Sobolev space  $W_4^{4,1}(\mathbb{R}^3)$ .*
- (B) *The positions of the oscillators  $\mathbf{a}_1, \dots, \mathbf{a}_N$  satisfy the two conditions:  
 $|\mathbf{a}_1| < |\mathbf{a}_2| < \dots < |\mathbf{a}_N|$  and  $\mathbf{a}_i \cdot \mathbf{a}_j \neq |\mathbf{a}_i| |\mathbf{a}_j|$ ,  $i \neq j$ .*

We notice that under assumption (A) the Hamiltonian  $H^\varepsilon$  is surely self-adjoint and bounded from below (in fact much less is required). In particular this implies that the unitary evolution of the system is well defined for any time  $t$ . Assumption (A) is also crucial for our specific method of proof, based on repeated integrations by parts in highly oscillatory integrals. Furthermore assumption (B) is a technical ingredient useful for the proof.

We stress that our model is completely defined by the Hamiltonian (2.10), the initial state (2.13) and the assumptions (A), (B). Before approaching the evolution problem, we briefly comment on the physical meaning of our scaling for  $\varepsilon \rightarrow 0$ .

We first observe that the dimensionless quantity

$$\frac{\hbar}{M v_0 \sigma} \quad (2.18)$$

is of order  $\varepsilon$ , which means that the wavelength  $\frac{\hbar}{M v_0}$  associated to the  $\alpha$ -particle is much smaller than the spatial localization  $\sigma$  (high momentum regime). Analogously for any  $j = 1, \dots, N$  the quantities

$$\frac{\sigma}{|\mathbf{a}_j|}, \quad \frac{\gamma}{|\mathbf{a}_j|}, \quad \frac{\delta}{|\mathbf{a}_j|} \quad (2.19)$$

are of order  $\varepsilon$ , i.e. the spatial localization of the test particle, the "diameter" of the oscillators and the range of the interaction are much smaller than the macroscopic distance  $|\mathbf{a}_j|$ .

Moreover we notice that the energy levels of the oscillators  $\hbar\omega(|n| + 3/2)$  are of order  $\varepsilon$  and the coupling constant  $\lambda$  is of order  $\varepsilon^2$  while the kinetic energy of the test particle is of order one for  $\varepsilon \rightarrow 0$ . This guarantees that the energy loss for the test particle due to the interaction with an oscillator is very small (quasi-elastic regime) and the perturbative approach can be used. Finally it is interesting to compare the characteristic times of the system. In particular we define the classical flight times for  $j = 1, \dots, N$  as the time spent by a classical particle, starting from the origin with velocity  $v_0$ , to reach the oscillator in  $\mathbf{a}_j$

$$\tau_j = \frac{|\mathbf{a}_j|}{v_0} \quad (2.20)$$

the "period" of the oscillators

$$T_o = \frac{2\pi}{\omega} \quad (2.21)$$

and the transit time, i.e. the time spent by the test particle to travel the diameter of an oscillator

$$T_t = \frac{\gamma}{v_0} \quad (2.22)$$

It turns out that

$$\frac{T_t}{T_o} = O(1) \quad (2.23)$$

i.e. the test particle can "see" the internal structure of the oscillators and

$$\frac{T_t}{\tau_j} = O(\varepsilon) \quad (2.24)$$

which implies that  $\tau_j$  can be reasonably identified as the collision time of the test particle with the oscillator in  $\mathbf{a}_j$ .

We are now ready to study the solution of the Schrödinger equation with initial datum  $\Psi_0^\varepsilon$

$$\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon, \quad \mathcal{U}^\varepsilon(t) = e^{-i\frac{t}{\varepsilon^2}H^\varepsilon} \quad (2.25)$$

for  $t > \tau_N$ , i.e. for a fixed time  $t$  sufficiently large so that all collisions of the test particle with the oscillators have taken place. In particular we shall perform a perturbative analysis computing the first correction for  $\varepsilon \rightarrow 0$  to the free evolution of the system

$$\mathcal{U}_0^\varepsilon(t)\Psi_0^\varepsilon, \quad \mathcal{U}_0^\varepsilon(t) = e^{-i\frac{t}{\varepsilon^2}H_0^\varepsilon} \quad (2.26)$$

In order to formulate our main result, it is convenient to introduce the following definition.

**Definition 2.1.** Let  $P_j^\varepsilon = P_j^\varepsilon(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N)$  be the following function

$$P_j^\varepsilon(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} P_{\underline{n},j}^\varepsilon(\mathbf{R}) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{k,k \neq j} \varphi_{0,k}^\varepsilon(\mathbf{r}_k) \quad (2.27)$$

where  $P_{\underline{n},j}^\varepsilon$  is the wave packet for the test particle given by

$$P_{\underline{n},j}^\varepsilon(\mathbf{R}) = \frac{C_{\underline{n},j}^\varepsilon}{\varepsilon^{3/2}} \mathcal{A}_{\underline{n},j} \left( \frac{\mathbf{R} - (\hat{\mathbf{a}}_j \cdot \mathbf{R}) \hat{\mathbf{a}}_j}{\varepsilon} \right) e^{-\frac{(\hat{\mathbf{a}}_j \cdot \mathbf{R} - Z_{\underline{n},j}^\varepsilon)^2}{2\varepsilon^2} + i \frac{\mathcal{V}_{\underline{n}}^\varepsilon}{\varepsilon^2} \hat{\mathbf{a}}_j \cdot \mathbf{R}} \quad (2.28)$$

$$C_{\underline{n},j}^\varepsilon = -\frac{i8\pi^{9/4}\mathcal{N}_\varepsilon}{v_0^3\tau_j^2} e^{\frac{i}{\varepsilon}|n|\tau_j + i\frac{|n|^2\tau_j}{2v_0^2}} \quad (2.29)$$

$$\mathcal{A}_{\underline{n},j}(\mathbf{y}) = e^{-i\frac{|\mathbf{y}|^2}{2\tau_j}} \left[ \widetilde{V} \cdot \widetilde{(\phi_{\underline{n}}\phi_{\underline{0}})} \right] (-\tau_j^{-1}\mathbf{y} + |n|v_0\hat{\mathbf{a}}_j) \quad (2.30)$$

$$Z_{\underline{n},j}^\varepsilon = \varepsilon \frac{|n|\tau_j}{v_0} \quad (2.31)$$

$$\mathcal{V}_{\underline{n}}^\varepsilon = v_0 - \varepsilon \frac{|n|}{v_0} \quad (2.32)$$

We underline that the wave packet  $P_{\underline{n},j}^\varepsilon$  plays a crucial role in our analysis. It is written as the product of two different wave packets. The first one is a function of the two dimensional vector  $\mathbf{R} - (\hat{\mathbf{a}}_j \cdot \mathbf{R})\hat{\mathbf{a}}_j$ , orthogonal to the direction of the  $j$ -th oscillator  $\hat{\mathbf{a}}_j$ , and it is well concentrated in position and momentum around the origin for  $\varepsilon \rightarrow 0$ . The second one is a one-dimensional gaussian wave packet in the variable  $\hat{\mathbf{a}}_j \cdot \mathbf{R}$ , i.e. a coordinate along the direction of the  $j$ -th oscillator  $\hat{\mathbf{a}}_j$ . For  $\varepsilon \rightarrow 0$  such wave packet is well concentrated in position around  $Z_{\underline{n},j}^\varepsilon$  and in momentum around  $\mathcal{V}_{\underline{n}}^\varepsilon$ .

With the notation introduced in definition 2.1, we state our main result.

**Theorem 2.1.** *Let us assume (A), (B). Then for any  $t > \tau_N$  there exists  $C(t) > 0$  such that*

$$\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon = \mathcal{U}_0^\varepsilon(t)\Psi_0^\varepsilon + \varepsilon^2 \sum_{j=1}^N \mathcal{U}_0^\varepsilon(t)P_j^\varepsilon + \mathcal{E}^\varepsilon(t) \quad (2.33)$$

where

$$\|\mathcal{E}^\varepsilon(t)\| \leq C(t) \|\widetilde{V}\|_{W_4^{4,1}} \varepsilon^3 \quad (2.34)$$

Let us conclude this section with few comments. Theorem 2.1 provides the required approximate dynamics of the system for  $\varepsilon$  small. Using the expressions for the free propagator  $\mathcal{U}_0(t)$ , the initial state  $\Psi_0^\varepsilon$  and the functions  $P_j^\varepsilon$ , formula (2.33) can be rewritten as

$$\begin{aligned} & (\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon)(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) \\ &= e^{-i\frac{t}{\varepsilon^2}NE_0^\varepsilon} \left[ \left( e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}\psi^\varepsilon \right)(\mathbf{R}) + \varepsilon^2 \sum_{j=1}^N \left( e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}P_{\underline{0},j}^\varepsilon \right)(\mathbf{R}) \right] \prod_{k=1}^N \varphi_{\underline{0},k}^\varepsilon(\mathbf{r}_k) \\ &+ \varepsilon^2 e^{-i\frac{t}{\varepsilon^2}(N-1)E_0^\varepsilon} \sum_{j=1}^N \sum_{\underline{n} \neq \underline{0}} e^{-i\frac{t}{\varepsilon^2}E_{\underline{n}}^\varepsilon} \left( e^{-i\frac{t}{\varepsilon^2}h_0^\varepsilon}P_{\underline{n},j}^\varepsilon \right)(\mathbf{R}) \varphi_{\underline{n},j}(\mathbf{r}_j) \prod_{k,k \neq j} \varphi_{\underline{0},k}(\mathbf{r}_k) \\ &+ \mathcal{E}^\varepsilon(t) \end{aligned} \quad (2.35)$$



In the above formula the approximate wave function of the system has been written as the sum of two terms (or histories). In the first one all oscillators remain in their ground state and the test particle is described by the free evolution of the spherical wave slightly deformed by the free evolution of the small wave packets  $P_{\underline{0},j}^\varepsilon$ ,  $j = 1, \dots, N$  emerging from each oscillator. The second term is a sum over  $j$ ,  $j = 1, \dots, N$ , where, in each term of the sum, one has only one oscillator in an excited state (say the  $j$ -th oscillator) and correspondingly the test particle described by the free evolution of the small wave packet  $P_{\underline{n},j}^\varepsilon$ ,  $\underline{n} \neq \underline{0}$ , emerging from the excited oscillator. We recall that the small wave packet emerging from the  $j$ -th oscillator is well concentrated in position and momentum. Therefore, under free evolution, it propagates along the direction  $\hat{\mathbf{a}}_j$  with momentum  $\mathcal{V}_{\underline{n}}^\varepsilon$ . We also notice that, for each  $j$ , the wave packet  $P_{\underline{0},j}^\varepsilon$  is produced by an elastic collision between the test particle (with momentum  $v_0$ ) and the  $j$ -th oscillator and therefore its momentum is unaffected ( $\mathcal{V}_{\underline{0}}^\varepsilon = v_0$ ). On the other hand, the wave packet  $P_{\underline{n},j}^\varepsilon$ ,  $\underline{n} \neq \underline{0}$ , is produced by an inelastic collision with energy loss  $\Delta E = \varepsilon|n|$ . Therefore, by the conservation of energy, its momentum is correctly given by  $\mathcal{V}_{\underline{n}}^\varepsilon$  at first order in  $\varepsilon$ . Finally, in order to make more transparent the structure of the wave packet emerging from an excited oscillator, we give an explicit computation in a specific case. In particular we consider the case of the oscillator in  $\hat{\mathbf{a}}_1$  excited to the state labeled by  $\underline{n}$  where, without loss of generality, we choose  $\hat{\mathbf{a}}_1 = (0, 0, 1)$ . Moreover we denote  $\mathbf{R} = (X, Y, Z)$  and use the shorthand notation  $C^\varepsilon = C_{\underline{n},1}^\varepsilon$ ,  $\mathcal{A}(X, Y) = \mathcal{A}_{\underline{n},1}(X, Y, 0)$ ,  $\mathcal{Z}^\varepsilon = \mathcal{Z}_{\underline{n},1}^\varepsilon$ ,  $\mathcal{V}^\varepsilon = \mathcal{V}_{\underline{n}}^\varepsilon$ . Then the wave packet  $P_{\underline{n},1}^\varepsilon$  is factorized into a function of the variables  $(X, Y)$ , depending on the product  $\tilde{V} \cdot \widetilde{\phi_{\underline{n}}\phi_{\underline{0}}}$ , and a gaussian in the variable  $Z$

$$P_{\underline{n},1}^\varepsilon(X, Y, Z) = \frac{C^\varepsilon}{\varepsilon^{3/2}} \mathcal{A}(\varepsilon^{-1}X, \varepsilon^{-1}Y) e^{-\frac{(Z-\mathcal{Z}^\varepsilon)^2}{2\varepsilon^2} + i\frac{\mathcal{V}^\varepsilon}{\varepsilon^2}Z} \quad (2.36)$$

with

$$\|P_{\underline{n},1}^\varepsilon\|_{L^2}^2 = \sqrt{\pi} |C^\varepsilon|^2 \int dX dY |\mathcal{A}(X, Y)|^2 \quad (2.37)$$

From (2.36), (2.37) one easily computes mean value and standard deviation for the position. In particular one finds that  $\langle X \rangle$ ,  $\langle Y \rangle$ ,  $\langle Z \rangle$ ,  $\Delta X$ ,  $\Delta Y$ ,  $\Delta Z$  are all  $O(\varepsilon)$ . In the Fourier space one has

$$\tilde{P}_{\underline{n},1}^\varepsilon(K_x, K_y, K_z) = \varepsilon^{3/2} C^\varepsilon \tilde{\mathcal{A}}(\varepsilon K_x, \varepsilon K_y) e^{-\frac{\varepsilon^2}{2}(K_z - \mathcal{V}^\varepsilon/\varepsilon^2)^2 - i\mathcal{Z}^\varepsilon K_z + \frac{i}{\varepsilon^2} \mathcal{Z}^\varepsilon \mathcal{V}^\varepsilon} \quad (2.38)$$

Therefore for the mean value of the momentum one has  $\langle P_x \rangle = O(\varepsilon)$ ,  $\langle P_y \rangle = O(\varepsilon)$ ,  $\langle P_z \rangle = \mathcal{V}^\varepsilon = v_0 + O(\varepsilon)$ , while for the standard deviation one finds that  $\Delta P_x$ ,  $\Delta P_y$ ,  $\Delta P_z$  are all  $O(\varepsilon)$ . The free evolution of the wave packet is also factorized into a free evolution of its part dependent on the variables  $(X, Y)$  and the free evolution of the gaussian part dependent on  $Z$ . Therefore, from the above computations, one can conclude that, for  $t$  not too large, the wave packet remains well concentrated in position and momentum and it propagates along the  $Z$ -axis with momentum approximately given by  $v_0$ . This means that for  $t = \tau_1$  the wave packet is concentrated around the oscillator in  $\mathbf{a}_1$  and for  $t > \tau_1$  it will continue its propagation along the  $Z$ -axis.

### 3. LINE OF THE PROOF

The proof of theorem 2.1 requires some intermediate results. In this section we describe the line of reasoning, we state without proof such intermediate results and we finally conclude with the proof of the theorem. The proof of the intermediate results will be given in sections 4, 5, 6, 7, 8. We start with Duhamel's formula to represent the solution of the Schrödinger equation

$$\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon = \mathcal{U}_0^\varepsilon(t)\Psi_0^\varepsilon - i \int_0^t ds \mathcal{U}^\varepsilon(t-s) V^\varepsilon \mathcal{U}_0^\varepsilon(s)\Psi_0^\varepsilon \quad (3.1)$$

Iterating twice we obtain

$$\mathcal{U}^\varepsilon(t)\Psi_0^\varepsilon = \mathcal{U}_0^\varepsilon(t) \left[ \Psi_0^\varepsilon + I^\varepsilon(t)\Psi_0^\varepsilon \right] + \mathcal{R}^\varepsilon(t) = \mathcal{U}_0^\varepsilon(t) \left[ \Psi_0^\varepsilon + \sum_{j=1}^N I_j^\varepsilon(t)\Psi_0^\varepsilon \right] + \mathcal{R}^\varepsilon(t) \quad (3.2)$$

where we have denoted

$$I^\varepsilon(t) = -i \int_0^t ds \mathcal{U}_0^\varepsilon(-s) V^\varepsilon \mathcal{U}_0^\varepsilon(s) \quad (3.3)$$

$$I_j^\varepsilon(t) = -i \int_0^t ds \mathcal{U}_0^\varepsilon(-s) V_j^\varepsilon \mathcal{U}_0^\varepsilon(s) \quad (3.4)$$

$$\mathcal{R}^\varepsilon(t) = \mathcal{U}_0^\varepsilon(t) J^\varepsilon(t) \Psi_0^\varepsilon - i \int_0^t ds \mathcal{U}^\varepsilon(t-s) V^\varepsilon \mathcal{U}_0^\varepsilon(s) J^\varepsilon(s) \Psi_0^\varepsilon \quad (3.5)$$

$$J^\varepsilon(t) = - \int_0^t ds \int_0^s d\sigma \mathcal{U}_0^\varepsilon(-s) V^\varepsilon \mathcal{U}_0^\varepsilon(s) \mathcal{U}_0^\varepsilon(-\sigma) V^\varepsilon \mathcal{U}_0^\varepsilon(\sigma) \quad (3.6)$$

In order to isolate the dominant term in  $I_j^\varepsilon(t)\Psi_0^\varepsilon$  for  $\varepsilon \rightarrow 0$  it is convenient to introduce some further notation. Let us consider the rotation matrix  $\mathcal{R}_j$ ,  $j = 1, \dots, N$ , defined by the condition

$$\mathcal{R}_j \hat{\mathbf{a}}_j = (0, 0, 1) \quad (3.7)$$

and the angle

$$\theta_0 = \frac{1}{2} \min_{j,k, j \neq k} \{ \theta_{jk}, \pi \}, \quad \theta_{jk} = \cos^{-1} (\hat{\mathbf{a}}_j \cdot \hat{\mathbf{a}}_k) \quad (3.8)$$

Furthermore we introduce the following two portions of the unit sphere  $S^2$  around the unit vectors  $(0, 0, 1)$  and  $\hat{\mathbf{a}}_j$  respectively

$$\mathcal{C}_0 = \{ \hat{\mathbf{u}} \in S^2 \mid \hat{u}_1^2 + \hat{u}_2^2 < \sin^2 \theta_0 \} \quad (3.9)$$

$$\mathcal{C}_j = \mathcal{R}_j \mathcal{C}_0 = \{ \hat{\mathbf{u}} \in S^2 \mid (\mathcal{R}_j \hat{\mathbf{u}})_1^2 + (\mathcal{R}_j \hat{\mathbf{u}})_2^2 < \sin^2 \theta_0 \} \quad (3.10)$$

We notice that  $\mathcal{C}_j \cap \mathcal{C}_k = \emptyset$ , for  $j \neq k$ ; moreover  $\hat{\mathbf{a}}_j \in \mathcal{C}_j$  and  $\hat{\mathbf{a}}_k \notin \mathcal{C}_j$  for  $j \neq k$ . Exploiting the above notation, we can define the "portion around  $\hat{\mathbf{a}}_j$ " of the initial spherical wave

$$\psi_j^\varepsilon(\mathbf{R}) = \frac{\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \eta(\varepsilon^{-1} \mathbf{R}) \int_{\mathcal{C}_j} d\hat{\mathbf{u}} e^{\frac{i}{\varepsilon^2} v_0 \hat{\mathbf{u}} \cdot \mathbf{R}} \quad (3.11)$$

and correspondingly

$$\Psi_{0,j}^\varepsilon(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \psi_j^\varepsilon(\mathbf{R}) \prod_{k=1}^N \varphi_{\mathbf{0},k}^\varepsilon(\mathbf{r}_k) \quad (3.12)$$

The portion of spherical wave defined in (3.11) has the following property: its classical evolution will hit the oscillator in  $\mathbf{a}_j$  and it will not hit the remaining oscillators in  $\mathbf{a}_k$ , for  $k \neq j$ . Taking into account definition (3.12), we rewrite (3.2) as follows

$$\mathcal{U}^\varepsilon(t) \Psi_0^\varepsilon = \mathcal{U}_0^\varepsilon(t) \left[ \Psi_0^\varepsilon + \sum_{j=1}^N I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon \right] + \sum_{j=1}^N \mathcal{U}_0^\varepsilon(t) I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon) + \mathcal{R}^\varepsilon(t) \quad (3.13)$$

The problem is now reduced to the analysis of the r.h.s. of (3.13) for  $\varepsilon \rightarrow 0$ . In particular, in order to prove theorem 2.1 we have to show that  $I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon = \varepsilon^2 P_j^\varepsilon + O(\varepsilon^3)$  and also that the last two terms of (3.13) are  $O(\varepsilon^3)$ .

The first step is to obtain a good representation of the relevant objects  $I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon$ ,  $I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)$  and  $J^\varepsilon(t) \Psi_0^\varepsilon$ . This is done in section 4, where for each object we perform an expansion in series of the eigenfunctions of the harmonic oscillators and the coefficients of the expansion are written as highly oscillatory integrals. Such representation formulas allow to exploit stationary and non stationary phase methods to characterize the asymptotic behavior of each object for  $\varepsilon \rightarrow 0$  ([F], [BH]). In the case of  $I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon$ , it turns out that the phase in the oscillatory integral has exactly one, non degenerate, critical point in the integration region if  $t > \tau_j$ . This is the crucial ingredient to prove

**Proposition 3.1.** *Let us assume (A), (B). Then for any  $t > \tau_j$  there exists  $C^1(t) > 0$  such that*

$$I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon = \varepsilon^2 P_j^\varepsilon + Q_j^\varepsilon(t) \quad (3.14)$$

where  $P_j^\varepsilon$  is given in definition 2.1 and

$$\|Q_j^\varepsilon(t)\| \leq C^1(t) \|\tilde{V}\|_{W_4^{4,1}} \varepsilon^3 \quad (3.15)$$

We shall prove decomposition (3.14), with an explicit expression for  $Q_j^\varepsilon(t)$ , in section 5, while the estimate (3.15) is proved in section 6.

For the estimate of  $I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)$  one exploits the fact that the corresponding phase in the oscillatory integral does not have critical points in the integration region. Therefore, by non stationary phase methods, in section 7 we prove the following proposition.

**Proposition 3.2.** *Let us assume  $\tilde{V} \in W_k^{k,1}(\mathbb{R}^3)$ ,  $k \in \mathbb{N}$ , and (B). Then for any  $t > 0$  there exists  $C_k^2(t) > 0$  such that*

$$\|I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)\| \leq C_k^2(t) \|\tilde{V}\|_{W_k^{k,1}} \varepsilon^{k-1} \quad (3.16)$$

The estimate of the rest  $\mathcal{R}^\varepsilon(t)$  is reduced to the estimate of  $J^\varepsilon(t) \Psi_0^\varepsilon$ . Exploiting the representation given in section 4, in section 8 we prove

**Proposition 3.3.** *Let us assume (A), (B). Then for any  $t > 0$  there exists  $C^3(t) > 0$  such that*

$$\|\mathcal{R}^\varepsilon(t)\| \leq (1 + t\|V\|_{L^\infty}) \sup_{s \leq t} \|J^\varepsilon(s)\Psi_0^\varepsilon\| \leq C^3(t) (1 + t\|V\|_{L^\infty}) \|\tilde{V}\|_{W_4^{4,1}} \varepsilon^3 \quad (3.17)$$

Collecting together the results stated in the above propositions we are in position to prove our main result.

*Proof of theorem 2.1.* The proof follows from formula (3.13), proposition 3.1, estimate (3.16) for  $k = 4$  and estimate (3.17).  $\square$

#### 4. REPRESENTATION FORMULAS

In this section we derive useful representation formulas for the relevant objects  $I_j^\varepsilon(t)\Psi_{0,j}^\varepsilon$ ,  $I_j^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)$  and  $J^\varepsilon(t)\Psi_0^\varepsilon$ . In particular we perform an expansion in eigenfunctions of the harmonic oscillators and we represent the corresponding Fourier coefficients as highly oscillatory integrals. Indeed, in terms of the rescaled variable

$$\mathbf{x} = \frac{\mathbf{R}}{\varepsilon} \quad (4.1)$$

we have

**Proposition 4.1.** *For any  $\varepsilon > 0$ ,  $\underline{n} \in \mathbb{N}^3$  and  $j = 1, \dots, N$  the following representation formulas hold*

$$(I_j^\varepsilon(t)\Psi_{0,j}^\varepsilon)(\varepsilon\mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} \mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{k, k \neq j} \varphi_{\underline{0},k}^\varepsilon(\mathbf{r}_k), \quad (4.2)$$

where

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) = -\frac{i\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int_0^t ds \int d\xi \int_{\mathcal{C}_j} d\hat{\mathbf{u}} F_{\underline{n}}(\xi, s; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_{\underline{n},j}(\xi, s, \hat{\mathbf{u}}; \mathbf{x})} \quad (4.3)$$

$$F_{\underline{n}}(\xi, s; \mathbf{x}) = e^{i\xi \cdot \mathbf{x} + i\frac{s}{2}\xi^2} g_{\underline{n},\underline{0}}(\xi) f(\mathbf{x} + s\xi) \quad (4.4)$$

$$g_{\underline{n},\underline{m}}(\xi) = \widetilde{\phi_{\underline{n}}\phi_{\underline{m}}}(\xi) \tilde{V}(\xi) \quad (4.5)$$

$$\Phi_{\underline{n},j}(\xi, s, \hat{\mathbf{u}}; \mathbf{x}) = -\xi \cdot \mathbf{a}_j + v_0 \hat{\mathbf{u}} \cdot (\mathbf{x} + s\xi) + |n|s \quad (4.6)$$

and

$$(I_j^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon))(\varepsilon\mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} \mathcal{T}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{i, i \neq j} \varphi_{\underline{0},i}^\varepsilon(\mathbf{r}_i), \quad (4.7)$$

where  $\mathcal{T}_{j,\underline{n}}^\varepsilon$  differs from  $\mathcal{I}_{j,\underline{n}}^\varepsilon$  only for the integration region, i.e.  $\mathcal{C}_j$  is replaced by  $S^2 \setminus \mathcal{C}_j$ .

*Proof.* We shall first give the representation formula for  $I_j^\varepsilon(t)\Psi_0^\varepsilon$ . From the definition (3.4) of  $I_j^\varepsilon(t)$  we have that

$$\begin{aligned}
& \int d\mathbf{r}_1 \dots d\mathbf{r}_N \varphi_{\underline{n}_1,1}^\varepsilon(\mathbf{r}_1) \dots \varphi_{\underline{n}_N,N}^\varepsilon(\mathbf{r}_N) (I_j^\varepsilon(t)\Psi_0^\varepsilon)(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) \\
&= -i \int_0^t ds e^{i\frac{s}{\varepsilon^2}E_{\underline{n}_1}^\varepsilon + \dots + i\frac{s}{\varepsilon^2}E_{\underline{n}_N}^\varepsilon - i\frac{s}{\varepsilon^2}E_{\underline{0}}^\varepsilon - \dots - i\frac{s}{\varepsilon^2}E_{\underline{0}}^\varepsilon} \int d\mathbf{r}_1 \dots d\mathbf{r}_N \varphi_{\underline{n}_1,1}^\varepsilon(\mathbf{r}_1) \dots \varphi_{\underline{n}_N,N}^\varepsilon(\mathbf{r}_N) \\
&\quad \cdot \left( e^{i\frac{s}{\varepsilon^2}h_0^\varepsilon} V_j^\varepsilon e^{-i\frac{s}{\varepsilon^2}h_0^\varepsilon} \psi^\varepsilon \varphi_{\underline{0},j}^\varepsilon \right)(\mathbf{R}, \mathbf{r}_j) \prod_{k,k \neq j} \varphi_{\underline{0},k}^\varepsilon(\mathbf{r}_k) \\
&= -i \prod_{k,k \neq j} \delta_{\underline{n}_k, \underline{0}} \int_0^t ds e^{i\frac{s}{\varepsilon} |n_j|} \int d\mathbf{r}_j \varphi_{\underline{n}_j,j}^\varepsilon(\mathbf{r}_j) \left( e^{i\frac{s}{\varepsilon^2}h_0^\varepsilon} V_j^\varepsilon e^{-i\frac{s}{\varepsilon^2}h_0^\varepsilon} \psi^\varepsilon \varphi_{\underline{0},j}^\varepsilon \right)(\mathbf{R}, \mathbf{r}_j) \quad (4.8)
\end{aligned}$$

Therefore

$$(I_j^\varepsilon(t)\Psi_0^\varepsilon)(\mathbf{R}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} \mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{R}) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{k,k \neq j} \varphi_{\underline{0},k}^\varepsilon(\mathbf{r}_k), \quad (4.9)$$

where

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{R}) = -i \int_0^t ds e^{i\frac{s}{\varepsilon} |n_j|} \left( e^{i\frac{s}{\varepsilon^2}h_0^\varepsilon} \langle \varphi_{\underline{n},j}^\varepsilon, V_j^\varepsilon \varphi_{\underline{0},j}^\varepsilon \rangle e^{-i\frac{s}{\varepsilon^2}h_0^\varepsilon} \psi^\varepsilon \right)(\mathbf{R}) \quad (4.10)$$

For the scalar product in  $L^2(\mathbb{R}^3)$  appearing in the integrand in (4.10) we have

$$\begin{aligned}
\langle \varphi_{\underline{n},j}^\varepsilon, V_j^\varepsilon \varphi_{\underline{0},j}^\varepsilon \rangle(\mathbf{R}) &= \frac{1}{\sqrt{\varepsilon^3}} \int d\mathbf{r}_j \phi_{\underline{n}}(\varepsilon^{-1}(\mathbf{r}_j - \mathbf{a}_j)) V(\varepsilon^{-1}(\mathbf{R} - \mathbf{r}_j)) \phi_{\underline{0}}(\varepsilon^{-1}(\mathbf{r}_j - \mathbf{a}_j)) \\
&= \int d\mathbf{x} \phi_{\underline{n}}(\mathbf{x}) \phi_{\underline{0}}(\mathbf{x}) V\left(\frac{\mathbf{R} - \mathbf{a}_j}{\varepsilon} - \mathbf{x}\right) = \int d\boldsymbol{\xi} \widetilde{\phi_{\underline{n}}\phi_{\underline{0}}}(\boldsymbol{\xi}) \tilde{V}(\boldsymbol{\xi}) e^{\frac{i}{\varepsilon}\boldsymbol{\xi} \cdot (\mathbf{R} - \mathbf{a}_j)} \\
&\equiv \int d\boldsymbol{\xi} g_{\underline{n},\underline{0}}(\boldsymbol{\xi}) e^{\frac{i}{\varepsilon}\boldsymbol{\xi} \cdot (\mathbf{R} - \mathbf{a}_j)} \quad (4.11)
\end{aligned}$$

hence

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{R}) = -i \int_0^t ds e^{i\frac{s}{\varepsilon} |n_j|} \int d\boldsymbol{\xi} g_{\underline{n},\underline{0}}(\boldsymbol{\xi}) e^{-\frac{i}{\varepsilon}\boldsymbol{\xi} \cdot \mathbf{a}_j} \left( e^{i\frac{s}{\varepsilon^2}h_0^\varepsilon} e^{\frac{i}{\varepsilon}\boldsymbol{\xi} \cdot (\cdot)} e^{-i\frac{s}{\varepsilon^2}h_0^\varepsilon} \psi^\varepsilon \right)(\mathbf{R}) \quad (4.12)$$

where  $e^{\frac{i}{\varepsilon}\xi \cdot (\cdot)}$  denotes the multiplication operator  $(e^{\frac{i}{\varepsilon}\xi \cdot (\cdot)}g)(\mathbf{R}) = e^{\frac{i}{\varepsilon}\xi \cdot \mathbf{R}}g(\mathbf{R})$ . Furthermore for any  $g \in L^2(\mathbb{R}^3)$  we have

$$\begin{aligned}
\left( e^{\frac{i}{\varepsilon^2}sh_0^\varepsilon} e^{\frac{i\xi}{\varepsilon} \cdot (\cdot)} e^{-\frac{i}{\varepsilon^2}sh_0^\varepsilon} g \right) (\mathbf{R}) &= \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{R}} \left( e^{\frac{i}{\varepsilon^2}sh_0^\varepsilon} e^{\frac{i\xi}{\varepsilon} \cdot (\cdot)} e^{-\frac{i}{\varepsilon^2}sh_0^\varepsilon} g \right)^\sim (\mathbf{k}) \\
&= \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{R}} e^{i\frac{\xi}{2}\mathbf{k}^2\varepsilon^2} \left( e^{\frac{i\xi}{\varepsilon} \cdot (\cdot)} e^{-\frac{i}{\varepsilon^2}sh_0^\varepsilon} g \right)^\sim (\mathbf{k}) \\
&= \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{R}} e^{i\frac{\xi}{2}\mathbf{k}^2\varepsilon^2} \left( e^{-\frac{i}{\varepsilon^2}sh_0^\varepsilon} g \right)^\sim \left( \mathbf{k} - \frac{\xi}{\varepsilon} \right) \\
&= \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{R}} e^{i\frac{\xi}{2}\mathbf{k}^2\varepsilon^2} e^{-i\frac{s\varepsilon^2}{2}(\mathbf{k} - \frac{\xi}{\varepsilon})^2} \tilde{g} \left( \mathbf{k} - \frac{\xi}{\varepsilon} \right) \\
&= \frac{e^{-i\frac{s\xi^2}{2}}}{\sqrt{(2\pi)^3}} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{R} + s\varepsilon\xi)} \tilde{g} \left( \mathbf{k} - \frac{\xi}{\varepsilon} \right) \\
&= e^{is\frac{\xi^2}{2}} e^{i\frac{\xi}{\varepsilon} \cdot \mathbf{R}} g(\mathbf{R} + \varepsilon s\xi)
\end{aligned} \tag{4.13}$$

Using (4.13) in (4.12) we have

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{R}) = -i \int_0^t ds \int d\xi g_{\underline{n},0}(\xi) e^{i\left(\frac{\xi}{\varepsilon}|n| - \frac{\xi}{\varepsilon} \cdot \mathbf{a}_j + s\frac{\xi^2}{2} + \frac{\xi}{\varepsilon} \cdot \mathbf{R}\right)} \psi^\varepsilon(\mathbf{R} + \varepsilon s\xi) \tag{4.14}$$

Substituting the explicit expression of  $\psi^\varepsilon$  in (4.14) one obtains the required representation of  $I_j^\varepsilon(t)\Psi_0^\varepsilon$ . The representation for  $I_j^\varepsilon(t)\Psi_{0,j}^\varepsilon$  and  $I_j^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)$  are obtained replacing in (4.14)  $\psi^\varepsilon$  with  $\psi_j^\varepsilon$  and  $\psi^\varepsilon - \psi_j^\varepsilon$  respectively.  $\square$

**Proposition 4.2.** *For any  $\varepsilon > 0$ ,  $\underline{n}, \underline{m} \in \mathbb{N}^3$  and  $k, l = 1, \dots, N$  we have the following representation formula*

$$J^\varepsilon(t) = \sum_{\substack{k,l=1 \\ k \neq l}}^N J_{k,l}^\varepsilon(t) + \sum_{k=1}^N J_k^\varepsilon(t) \tag{4.15}$$

$$J_{k,l}^\varepsilon(t) = - \int_0^t ds \int_0^s d\sigma \mathcal{U}_0^\varepsilon(-s) V_k^\varepsilon \mathcal{U}_0^\varepsilon(s) \mathcal{U}_0^\varepsilon(-\sigma) V_l^\varepsilon \mathcal{U}_0^\varepsilon(\sigma), \quad J_k^\varepsilon(t) \equiv J_{k,k}^\varepsilon(t) \tag{4.16}$$

$$(J_k^\varepsilon(t)\Psi_0^\varepsilon)(\varepsilon\mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}, \underline{m}} \mathcal{J}_{\underline{n}, \underline{m}}^{\varepsilon, k}(t, \mathbf{x}) \varphi_{\underline{n}, k}^\varepsilon(\mathbf{r}_k) \prod_{i, i \neq k} \varphi_{0, i}^\varepsilon(\mathbf{r}_i) \tag{4.17}$$

$$(J_{k,l}^\varepsilon(t)\Psi_0^\varepsilon)(\varepsilon\mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}, \underline{m}} \mathcal{J}_{\underline{n}, \underline{m}}^{\varepsilon, k, l}(t, \mathbf{x}) \varphi_{\underline{n}, k}^\varepsilon(\mathbf{r}_k) \varphi_{\underline{m}, l}^\varepsilon(\mathbf{r}_l) \prod_{i, i \neq k, l} \varphi_{0, i}^\varepsilon(\mathbf{r}_i) \tag{4.18}$$

where

$$\mathcal{J}_{\underline{n},\underline{m}}^{\varepsilon,k}(t, \mathbf{x}) = -\frac{\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{S^2} d\hat{\mathbf{u}} L_{\underline{n},\underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n},\underline{m}}^k(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x})} \quad (4.19)$$

$$\mathcal{J}_{\underline{n},\underline{m}}^{\varepsilon,k,l}(t, \mathbf{x}) = -\frac{\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{S^2} d\hat{\mathbf{u}} G_{\underline{n},\underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n},\underline{m}}^{k,l}(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x})} \quad (4.20)$$

$$L_{\underline{n},\underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) = e^{i(\eta+\xi)\cdot\mathbf{x}+i(\frac{s}{2}\eta^2+\frac{\sigma}{2}\xi^2)+is\eta\cdot\xi} g_{\underline{n},\underline{m}}(\eta) g_{\underline{m},\underline{0}}(\xi) f(\mathbf{x} + \sigma \xi + s \eta) \quad (4.21)$$

$$G_{\underline{n},\underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) = e^{i(\eta+\xi)\cdot\mathbf{x}+i(\frac{s}{2}\eta^2+\frac{\sigma}{2}\xi^2)+is\eta\cdot\xi} g_{\underline{n},\underline{0}}(\eta) g_{\underline{m},\underline{0}}(\xi) f(\mathbf{x} + \sigma \xi + s \eta) \quad (4.22)$$

$$\Theta_{\underline{n},\underline{m}}^k(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x}) = -(\xi + \eta) \cdot \mathbf{a}_k + v_0 \hat{\mathbf{u}} \cdot (\mathbf{x} + \sigma \xi + s \eta) + (|n| - |m|)s + |m|\sigma \quad (4.23)$$

$$\Theta_{\underline{n},\underline{m}}^{k,l}(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x}) = -\xi \cdot \mathbf{a}_l - \eta \cdot \mathbf{a}_k + v_0 \hat{\mathbf{u}} \cdot (\mathbf{x} + \sigma \xi + s \eta) + |n|s + |m|\sigma \quad (4.24)$$

The proof proceeds along the same line of the previous one and it is omitted for the sake of brevity.

## 5. DERIVATION OF THE LEADING TERM

In this section we analyze the term  $I_j^\varepsilon(t) \Psi_{0,j}^\varepsilon$ . In particular, we shall prove decomposition (3.14), with an explicit expression for  $Q_j^\varepsilon(t)$ . We start from formulas (4.2), (4.3) proved in proposition 4.1. In particular, by the change of variables  $\xi \rightarrow \mathcal{R}_j \xi$ ,  $\hat{\mathbf{u}} \rightarrow \mathcal{R}_j \hat{\mathbf{u}}$ , with  $\mathcal{R}_j$  defined in (3.7), we obtain

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) = -\frac{i \mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int_0^t ds \int d\xi \int_{\mathcal{C}_0} d\hat{\mathbf{u}} F_{\underline{n},j}^0(\xi, s; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_{\underline{n},j}^0(\xi, s, \hat{\mathbf{u}}; \mathbf{x})} \quad (5.1)$$

where

$$F_{\underline{n},j}^0(\xi, s; \mathbf{x}) \equiv F_{\underline{n}}(\mathcal{R}_j^{-1} \xi, s; \mathbf{x}) = e^{i\xi \cdot \mathbf{x}^j + i\frac{s}{2}\xi^2} g_{\underline{n},0}(\mathcal{R}_j^{-1} \xi) f(\mathbf{x}^j + s \xi) \quad (5.2)$$

$$\Phi_{\underline{n},j}^0(\xi, s, \hat{\mathbf{u}}; \mathbf{x}) \equiv \Phi_{\underline{n},j}(\mathcal{R}_j^{-1} \xi, s, \mathcal{R}_j^{-1} \hat{\mathbf{u}}; \mathbf{x}) = -|a_j| \xi_3 + v_0 \hat{\mathbf{u}} \cdot (\mathbf{x}^j + s \xi) + |n|s \quad (5.3)$$

$$\mathbf{x}^j \equiv \mathcal{R}_j \mathbf{x} \quad (5.4)$$

Notice that

$$\mathbf{x}_3^j = (0, 0, 1) \cdot \mathcal{R}_j \mathbf{x} = \hat{\mathbf{a}}_j \cdot \mathbf{x} \quad (5.5)$$

$$(\mathbf{x}_1^j, \mathbf{x}_2^j, 0) = \mathbf{x}^j - (\hat{\mathbf{a}}_j \cdot \mathbf{x})(0, 0, 1) = \mathcal{R}_j (\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j) \quad (5.6)$$

Moreover we parametrize the unit vector  $\hat{\mathbf{u}} \in \mathcal{C}_0$  as follows

$$\hat{\mathbf{u}} = \left( \mu, \nu, \sqrt{1 - \mu^2 - \nu^2} \right), \quad (\mu, \nu) \in D_0 \equiv \{ (a, b) \in \mathbb{R}^2, a^2 + b^2 < \sin^2 \theta_0 \} \quad (5.7)$$

Therefore the integral (5.1) is rewritten as

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) = -\frac{i \mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int d\xi \int_0^t ds \int_{D_0} d\mu d\nu F_{\underline{n},j}^1(\xi, \mu, \nu, s; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_{\underline{n},j}^1(\xi, \mu, \nu, s; \mathbf{x})} \quad (5.8)$$

where

$$F_{\underline{n},j}^1(\boldsymbol{\xi}, \mu, \nu, s; \mathbf{x}) \equiv \frac{1}{\sqrt{1-\mu^2-\nu^2}} F_{\underline{n},j}^0(\boldsymbol{\xi}, s; \mathbf{x}) = \frac{1}{\sqrt{1-\mu^2-\nu^2}} e^{i\boldsymbol{\xi} \cdot \mathbf{x}^j + i\frac{s}{2}\boldsymbol{\xi}^2} g_{\underline{n},0}(\mathcal{R}_j^{-1}\boldsymbol{\xi}) f(\mathbf{x}^j + s\boldsymbol{\xi}) \quad (5.9)$$

$$\Phi_{\underline{n},j}^1(\boldsymbol{\xi}, \mu, \nu, s; \mathbf{x}) = -|\mathbf{a}_j| \xi_3 + v_0 \left[ \mu(\mathbf{x}_1^j + s\xi_1) + \nu(\mathbf{x}_2^j + s\xi_2) + \sqrt{1-\mu^2-\nu^2}(\mathbf{x}_3^j + s\xi_3) \right] + |n|s \quad (5.10)$$

By a straightforward computation one can verify that the phase  $\Phi_{\underline{n},j}^1$  has exactly one (non degenerate) critical point in the integration region given by  $\boldsymbol{\xi} = \boldsymbol{\xi}^c$ ,  $(\mu, \nu, s) = \mathbf{z}^c$ , where

$$\boldsymbol{\xi}^c = (-\tau_j^{-1}x_1^j, -\tau_j^{-1}x_2^j, -v_0^{-1}|n|), \quad \mathbf{z}^c = (0, 0, \tau_j) \quad (5.11)$$

Therefore by stationary phase methods we can compute the asymptotic expansion of the oscillatory integral for  $\varepsilon \rightarrow 0$ . In the specific case, we can exploit the linearity of the phase in the variable  $\boldsymbol{\xi}$  to obtain the expansion in a relatively direct way. In particular we write

$$\Phi_{\underline{n},j}^1(\boldsymbol{\xi}, \mu, \nu, s; \mathbf{x}) = \mathbf{A}_j(\mu, \nu, s) \cdot \boldsymbol{\xi} + B_{\underline{n},j}(\mu, \nu, s; \mathbf{x}) \quad (5.12)$$

$$\mathbf{A}_j(\mu, \nu, s) = v_0 \left( \mu s, \nu s, \sqrt{1-\mu^2-\nu^2} s - \tau_j \right) \quad (5.13)$$

$$B_{\underline{n},j}(\mu, \nu, s; \mathbf{x}) = v_0 x_1^j \mu + v_0 x_2^j \nu + v_0 x_3^j \sqrt{1-\mu^2-\nu^2} + |n|s \quad (5.14)$$

and

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) = -\frac{i\mathcal{N}_\varepsilon}{\varepsilon^{5/2}} \int_0^t ds \int_{D_0} d\mu d\nu e^{\frac{i}{\varepsilon} B_{\underline{n},j}(\mu, \nu, s; \mathbf{x})} \int d\boldsymbol{\xi} F_{\underline{n},j}^1(\boldsymbol{\xi}, \mu, \nu, s; \mathbf{x}) e^{\frac{i}{\varepsilon} \mathbf{A}_j(\mu, \nu, s) \cdot \boldsymbol{\xi}} \quad (5.15)$$

Let us introduce the following linear change of coordinates

$$(\mu, \nu, s) = L_\varepsilon(z_1, z_2, z_3) \equiv L_\varepsilon \mathbf{z} \quad (5.16)$$

$$\mu = \frac{\varepsilon}{v_0 \tau_j} z_1, \quad \nu = \frac{\varepsilon}{v_0 \tau_j} z_2, \quad s = \tau_j + \frac{\varepsilon}{v_0} z_3 \quad (5.17)$$

The domain of integration in the variable  $\mathbf{z}$  is

$$\Omega_\varepsilon = \left\{ \mathbf{z} \in \mathbb{R}^3 \mid z_1^2 + z_2^2 < \varepsilon^{-2} v_0^2 \tau_j^2 \sin^2 \theta_0, \quad -\varepsilon^{-1} v_0 \tau_j < z_3 < \varepsilon^{-1} v_0 (t - \tau_j) \right\} \quad (5.18)$$

We notice that for  $\varepsilon \rightarrow 0$  one has  $L_\varepsilon \mathbf{z} \rightarrow \mathbf{z}^c$ , and  $\Omega_\varepsilon \rightarrow \mathbb{R}^3$ . In the new integration variables  $\mathbf{z}$  the integral (5.15) reads

$$\mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) = -\frac{i\mathcal{N}_\varepsilon \sqrt{\varepsilon}}{v_0^3 \tau_j^2} \int_{\Omega_\varepsilon} d\mathbf{z} e^{\frac{i}{\varepsilon} B_{\underline{n},j}(L_\varepsilon \mathbf{z}; \mathbf{x})} \int d\boldsymbol{\xi} F_{\underline{n},j}^1(\boldsymbol{\xi}, L_\varepsilon \mathbf{z}; \mathbf{x}) e^{\frac{i}{\varepsilon} \mathbf{A}_j(L_\varepsilon \mathbf{z}) \cdot \boldsymbol{\xi}} \quad (5.19)$$



Let us expand  $\mathbf{A}_j(L_\varepsilon \mathbf{z})$  and  $B_{\underline{n},j}(L_\varepsilon \mathbf{z}; \mathbf{x})$  around  $\varepsilon = 0$

$$\begin{aligned} \mathbf{A}_j(L_\varepsilon \mathbf{z}) &= \left( \varepsilon z_1 + \frac{\varepsilon^2}{v_0 \tau_j} z_1 z_3, \varepsilon z_2 + \frac{\varepsilon^2}{v_0 \tau_j} z_2 z_3, \varepsilon z_3 + \left( \sqrt{1 - \frac{\varepsilon^2}{v_0^2 \tau_j^2} (z_1^2 + z_2^2)} - 1 \right) (v_0 \tau_j + \varepsilon z_3) \right) \\ &= \varepsilon \mathbf{z} + \varepsilon^2 \left( \frac{z_1 z_3}{v_0 \tau_j}, \frac{z_2 z_3}{v_0 \tau_j}, \varepsilon^{-2} \left( \sqrt{1 - \frac{\varepsilon^2}{v_0^2 \tau_j^2} (z_1^2 + z_2^2)} - 1 \right) (v_0 \tau_j + \varepsilon z_3) \right) \end{aligned} \quad (5.20)$$

$$\equiv \varepsilon \mathbf{z} + \varepsilon^2 \mathbf{A}_{j,\varepsilon}^2(\mathbf{z}) \quad (5.21)$$

$$\begin{aligned} B_{\underline{n},j}(L_\varepsilon \mathbf{z}; \mathbf{x}) &= v_0 x_3^j + |n| \tau_j + \varepsilon \frac{x_1^j}{\tau_j} z_1 + \varepsilon \frac{x_2^j}{\tau_j} z_2 + \varepsilon \frac{|n|}{v_0} z_3 + v_0 x_3^j \left( \sqrt{1 - \frac{\varepsilon^2}{v_0^2 \tau_j^2} (z_1^2 + z_2^2)} - 1 \right) \\ &\equiv v_0 \hat{\mathbf{a}}_j \cdot \mathbf{x} + |n| \tau_j - \varepsilon \boldsymbol{\xi}^c \cdot \mathbf{z} + \varepsilon^2 B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x}) \end{aligned} \quad (5.22)$$

$$\equiv v_0 \hat{\mathbf{a}}_j \cdot \mathbf{x} + |n| \tau_j - \varepsilon \boldsymbol{\xi}^c \cdot \mathbf{z} + \varepsilon^2 B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x}) \quad (5.23)$$

where we have used  $x_3^j = (0, 0, 1) \cdot \mathcal{R}_j \mathbf{x} = \hat{\mathbf{a}}_j \cdot \mathbf{x}$  and (5.11). Correspondingly, the integral (5.19) can be written as

$$\begin{aligned} \mathcal{I}_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) &= -\frac{i \mathcal{N}_\varepsilon \sqrt{\varepsilon}}{v_0^3 \tau_j^2} e^{\frac{i}{\varepsilon} (v_0 \hat{\mathbf{a}}_j \cdot \mathbf{x} + |n| \tau_j)} \int_{\Omega_\varepsilon} d\mathbf{z} e^{-i \boldsymbol{\xi}^c \cdot \mathbf{z} + i \varepsilon B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x})} \int d\boldsymbol{\xi} F_{\underline{n},j}^1(\boldsymbol{\xi}, L_\varepsilon \mathbf{z}; \mathbf{x}) e^{i \mathbf{z} \cdot \boldsymbol{\xi} + i \varepsilon \mathbf{A}_{j,\varepsilon}^2(\mathbf{z})} \\ &\equiv -\frac{i \mathcal{N}_\varepsilon \sqrt{\varepsilon}}{v_0^3 \tau_j^2} e^{\frac{i}{\varepsilon} (v_0 \hat{\mathbf{a}}_j \cdot \mathbf{x} + |n| \tau_j)} (2\pi)^3 F_{\underline{n},j}^1(\boldsymbol{\xi}^c, \mathbf{z}^c; \mathbf{x}) + Q_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) \end{aligned} \quad (5.24)$$

where

$$\begin{aligned} Q_{j,\underline{n}}^\varepsilon(t, \mathbf{x}) &= -\frac{i \mathcal{N}_\varepsilon \sqrt{\varepsilon}}{v_0^3 \tau_j^2} e^{\frac{i}{\varepsilon} (v_0 \hat{\mathbf{a}}_j \cdot \mathbf{x} + |n| \tau_j)} \left[ \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z} e^{-i \boldsymbol{\xi}^c \cdot \mathbf{z}} \int d\boldsymbol{\xi} e^{i \mathbf{z} \cdot \boldsymbol{\xi}} F_{\underline{n},j}^1(\boldsymbol{\xi}, \mathbf{z}^c; \mathbf{x}) \right. \\ &\quad \left. + \int_{\Omega_\varepsilon} d\mathbf{z} e^{-i \boldsymbol{\xi}^c \cdot \mathbf{z}} \int d\boldsymbol{\xi} e^{i \mathbf{z} \cdot \boldsymbol{\xi}} \left( F_{\underline{n},j}^1(\boldsymbol{\xi}, L_\varepsilon \mathbf{z}; \mathbf{x}) e^{i \varepsilon (B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x}) + \mathbf{A}_{j,\varepsilon}^2(\mathbf{z}))} - F_{\underline{n},j}^1(\boldsymbol{\xi}, \mathbf{z}^c; \mathbf{x}) \right) \right] \end{aligned} \quad (5.25)$$

Let us compute  $F_{\underline{n},j}^1(\boldsymbol{\xi}^c, \mathbf{z}^c; \mathbf{x})$  (see (5.9)). Taking into account that  $\mathbf{z}^c = (0, 0, \tau_j)$  and

$$\boldsymbol{\xi}^c = -\tau_j^{-1} (x_1^j, x_2^j, 0) - |n| v_0^{-1} (0, 0, 1) = -\tau_j^{-1} \mathcal{R}_j (\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j) - |n| v_0^{-1} \mathcal{R}_j \hat{\mathbf{a}}_j \quad (5.26)$$

we have

$$\begin{aligned} F_{\underline{n},j}^1(\boldsymbol{\xi}^c, \mathbf{z}^c; \mathbf{x}) &= e^{i \boldsymbol{\xi}^c \cdot \mathbf{x}^j + \frac{i}{2} \tau_j (\boldsymbol{\xi}^c)^2} g_{\underline{n},j}(\mathcal{R}_j^{-1} \boldsymbol{\xi}^c) f(\mathbf{x}^j + \tau_j \boldsymbol{\xi}^c) \\ &= e^{i \left[ -\frac{1}{\tau_j} (\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j) \right] \cdot \mathbf{x} - i \frac{|n|}{v_0} \hat{\mathbf{a}}_j \cdot \mathbf{x} + i \frac{\tau_j}{2} \left( \frac{1}{\tau_j^2} |\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j|^2 + \frac{|n|^2}{v_0^2} \right)} g_{\underline{n},j} \left( -\tau_j^{-1} (\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j) - |n| v_0^{-1} \hat{\mathbf{a}}_j \right) \\ &\quad \cdot f(\hat{\mathbf{a}}_j \cdot \mathbf{x} - |n| v_0^{-1} \tau_j) \\ &= e^{i \frac{|n|^2 \tau_j}{2 v_0^2} - i \frac{|n|}{v_0} \hat{\mathbf{a}}_j \cdot \mathbf{x} - i \frac{|\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j|^2}{2 \tau_j}} g_{\underline{n},j} \left( -\tau_j^{-1} (\mathbf{x} - (\hat{\mathbf{a}}_j \cdot \mathbf{x}) \hat{\mathbf{a}}_j) - |n| v_0^{-1} \hat{\mathbf{a}}_j \right) f(\hat{\mathbf{a}}_j \cdot \mathbf{x} - |n| v_0^{-1} \tau_j) \end{aligned} \quad (5.27)$$

From (5.24), (5.25), (5.27) and definition 2.1 we obtain

$$\mathcal{I}_{\underline{n},j}^\varepsilon(t, \mathbf{x}) = \varepsilon^2 P_{\underline{n},j}^\varepsilon(\varepsilon \mathbf{x}) + Q_{\underline{n},j}^\varepsilon(t, \mathbf{x}) \quad (5.28)$$

Taking into account of (5.28) and (4.2), we have proved the decomposition (3.14), with  $Q_j^\varepsilon(t)$  explicitly given by

$$Q_j^\varepsilon(t, \mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} Q_{\underline{n},j}^\varepsilon(t, \mathbf{x}) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{k, k \neq j} \varphi_{0,k}^\varepsilon(\mathbf{r}_k) \quad (5.29)$$

## 6. ESTIMATE OF $Q_j^\varepsilon(t)$

Let us first recall some elementary facts that will be used in this and in the following sections. The unitary propagator  $U(t)$  of the harmonic oscillator centered in the origin and corresponding to  $\hbar = m = \omega = 1$  has an integral kernel given by

$$U(t, \mathbf{x}, \mathbf{y}) = \sum_{\underline{n}} \phi_{\underline{n}}(\mathbf{x}) \phi_{\underline{n}}(\mathbf{y}) e^{-it(|\underline{n}| + \frac{3}{2})} \quad (6.1)$$

Using such representation we can compute the following sum

$$\begin{aligned} & \sum_{\underline{n}} e^{-i\frac{|\underline{n}|}{\varepsilon}t} g_{\underline{n},0}(\boldsymbol{\xi}) \bar{g}_{\underline{n},0}(\boldsymbol{\xi}') = \\ & \tilde{V}(\boldsymbol{\xi}) \bar{\tilde{V}}(\boldsymbol{\xi}') \frac{1}{(2\pi)^3} \sum_{\underline{n}} \int d\mathbf{y} \phi_{\underline{n}}(\mathbf{y}) \phi_{\underline{0}}(\mathbf{y}) e^{-i\mathbf{y} \cdot \boldsymbol{\xi}} \int d\mathbf{y}' \phi_{\underline{n}}(\mathbf{y}') \phi_{\underline{0}}(\mathbf{y}') e^{i\mathbf{y}' \cdot \boldsymbol{\xi}'} e^{-i\frac{|\underline{n}|}{\varepsilon}t} = \\ & \tilde{V}(\boldsymbol{\xi}) \bar{\tilde{V}}(\boldsymbol{\xi}') \frac{1}{(2\pi)^3} \int d\mathbf{y}' \overline{\phi_{\underline{0}}(\mathbf{y}') e^{-i\mathbf{y}' \cdot \boldsymbol{\xi}'}} \int d\mathbf{y} \sum_{\underline{n}} \phi_{\underline{n}}(\mathbf{y}) \phi_{\underline{n}}(\mathbf{y}') e^{-i\frac{|\underline{n}|}{\varepsilon}t} \phi_{\underline{0}}(\mathbf{y}) e^{-i\mathbf{y} \cdot \boldsymbol{\xi}} = \\ & \tilde{V}(\boldsymbol{\xi}) \bar{\tilde{V}}(\boldsymbol{\xi}') \frac{e^{i\frac{3}{2\varepsilon}t}}{(2\pi)^3} \langle \phi_{\underline{0}}, e^{i\boldsymbol{\xi}' \cdot (\cdot)} U(t/\varepsilon) e^{-i\boldsymbol{\xi} \cdot (\cdot)} \phi_{\underline{0}} \rangle \end{aligned} \quad (6.2)$$

In particular for  $t = 0$  one has

$$\sum_{\underline{n}} g_{\underline{n},0}(\boldsymbol{\xi}) \bar{g}_{\underline{n},0}(\boldsymbol{\xi}') = \frac{\tilde{V}(\boldsymbol{\xi}) \bar{\tilde{V}}(\boldsymbol{\xi}')}{(2\pi)^3} e^{-\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')^2}{4}} \quad (6.3)$$

Let us now prove the estimate (3.15). Taking into account of (5.29) and (5.25) we write

$$Q_j^\varepsilon(t)(t, \mathbf{x}, \mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{\underline{n}} (Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x}) + Q_{j,\underline{n}}^{\varepsilon,2}(t, \mathbf{x})) \varphi_{\underline{n},j}^\varepsilon(\mathbf{r}_j) \prod_{k, k \neq j} \varphi_{0,k}^\varepsilon(\mathbf{r}_k), \quad (6.4)$$

where

$$Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x}) = -\frac{i\mathcal{N}_\varepsilon\sqrt{\varepsilon}}{v_0^3\tau_j^2} e^{\frac{i}{\varepsilon}(v_0\hat{\mathbf{a}}_j\cdot\mathbf{x}+|n|\tau_j)} \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} d\mathbf{z} e^{-i\xi^c\cdot\mathbf{z}} \int d\xi e^{i\mathbf{z}\cdot\xi} F_{\underline{n},j}^1(\xi, \mathbf{z}^c; \mathbf{x}) \quad (6.5)$$

$$Q_{j,\underline{n}}^{\varepsilon,2}(t, \mathbf{x}) = -\frac{i\mathcal{N}_\varepsilon\sqrt{\varepsilon}}{v_0^3\tau_j^2} e^{\frac{i}{\varepsilon}(v_0\hat{\mathbf{a}}_j\cdot\mathbf{x}+|n|\tau_j)} \int_{\Omega_\varepsilon} d\mathbf{z} e^{-i\xi^c\cdot\mathbf{z}} \int d\xi e^{i\mathbf{z}\cdot\xi} \cdot \left( F_{\underline{n},j}^1(\xi, L_\varepsilon\mathbf{z}; \mathbf{x}) e^{i\varepsilon(B_{j,\varepsilon}^2(\mathbf{z};\mathbf{x})+A_{j,\varepsilon}^2(\mathbf{z}))} - F_{\underline{n},j}^1(\xi, \mathbf{z}^c; \mathbf{x}) \right) \quad (6.6)$$

and, from (5.9),(5.11),(5.16),(5.17), the explicit expressions for  $F_{\underline{n},j}^1(\xi, \mathbf{z}^c; \mathbf{x})$  and  $F_{\underline{n},j}^1(\xi, L_\varepsilon\mathbf{z}; \mathbf{x})$  are given by

$$F_{\underline{n},j}^1(\xi, L_\varepsilon\mathbf{z}; \mathbf{x}) = \frac{1}{\sqrt{1 - \frac{\varepsilon^2}{v_0^2\tau_j^2}(z_1^2+z_2^2)}} e^{i\xi\cdot\mathbf{x}^j + i\frac{\tau_j}{2}\xi^2 + i\varepsilon\frac{z_3}{2v_0}\xi^2} g_{\underline{n},0}(\mathcal{R}_j^{-1}\xi) f\left(\mathbf{x}^j + \tau_j\xi + \varepsilon\frac{z_3}{v_0}\xi\right) \quad (6.7)$$

$$F_{\underline{n},j}^1(\xi, \mathbf{z}^c; \mathbf{x}) = e^{i\xi\cdot\mathbf{x}^j + i\frac{\tau_j}{2}\xi^2} g_{\underline{n},0}(\mathcal{R}_j^{-1}\xi) f\left(\mathbf{x}^j + \tau_j\xi\right) \quad (6.8)$$

From (6.4) we have

$$\|Q_j^\varepsilon(t)\|^2 \leq 2\varepsilon^3 \sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x})|^2 + 2\varepsilon^3 \sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,2}(t, \mathbf{x})|^2 \quad (6.9)$$

Let us estimate the first term in the r.h.s of (6.9). Using (6.5) and (6.8) we have

$$\begin{aligned} & \sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x})|^2 \\ &= \frac{\mathcal{N}_\varepsilon^2\varepsilon}{v_0^6\tau_j^4} \sum_{\underline{n}} \int d\mathbf{x} \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} d\mathbf{z} e^{-i\xi^c\cdot\mathbf{z}} \int d\xi e^{i\mathbf{z}\cdot\xi} e^{i\xi\cdot\mathbf{x}^j + i\frac{\tau_j}{2}\xi^2} g_{\underline{n},0}(\mathcal{R}_j^{-1}\xi) f(\mathbf{x}^j + \tau_j\xi) \\ & \cdot \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} d\mathbf{z}' e^{i\xi^c\cdot\mathbf{z}'} \int d\xi' e^{-i\mathbf{z}'\cdot\xi'} e^{-i\xi'\cdot\mathbf{x}^j - i\frac{\tau_j}{2}\xi'^2} \bar{g}_{\underline{n},0}(\mathcal{R}_j^{-1}\xi') f(\mathbf{x}^j + \tau_j\xi') \\ &= \frac{\mathcal{N}_\varepsilon^2\varepsilon}{v_0^6\tau_j^4(2\pi)^3} \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} d\mathbf{z} \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} d\mathbf{z}' \int d\mathbf{x} e^{-i\xi^c\cdot(\mathbf{z}-\mathbf{z}')} \int d\xi \int d\xi' e^{i\mathbf{z}\cdot\xi - i\mathbf{z}'\cdot\xi'} e^{i(\xi-\xi')\cdot\mathbf{x}^j + i\frac{\tau_j}{2}(\xi^2-\xi'^2)} \\ & \cdot \tilde{V}(\mathcal{R}_j^{-1}\xi) \bar{\tilde{V}}(\mathcal{R}_j^{-1}\xi') e^{-\frac{(\xi-\xi')^2}{4}} f(\mathbf{x}^j + \tau_j\xi) f(\mathbf{x}^j + \tau_j\xi') \end{aligned} \quad (6.10)$$

where the sum over  $\underline{n}$  in (6.10) has been explicitly computed exploiting (6.3). Using the identity

$$e^{i\mathbf{z}\cdot\xi} = -\frac{1}{|\mathbf{z}|^2} \Delta_\xi e^{i\mathbf{z}\cdot\xi} = \frac{1}{|\mathbf{z}|^4} (\Delta_\xi)^2 e^{i\mathbf{z}\cdot\xi} \quad (6.11)$$

and integrating by parts we find

$$\begin{aligned}
& \sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x})|^2 \\
&= \frac{\mathcal{N}_\varepsilon^2 \varepsilon}{v_0^6 \tau_j^4 (2\pi)^3} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z} \frac{1}{|\mathbf{z}|^4} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z}' \frac{1}{|\mathbf{z}'|^4} \int d\mathbf{x} e^{-i\boldsymbol{\xi}^c \cdot (\mathbf{z} - \mathbf{z}')} \int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' e^{i\mathbf{z} \cdot \boldsymbol{\xi} - i\mathbf{z}' \cdot \boldsymbol{\xi}'} (\Delta_{\boldsymbol{\xi}})^2 (\Delta_{\boldsymbol{\xi}'} )^2 \\
&\quad \left( \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi}) \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi}') e^{-\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')^2}{4}} e^{i(\boldsymbol{\xi} \cdot \mathbf{x}^j + \frac{\tau_j}{2} \boldsymbol{\xi}^2)} f(\mathbf{x}^j + \tau_j \boldsymbol{\xi}) e^{-i(\boldsymbol{\xi}' \cdot \mathbf{x}^j + \frac{\tau_j}{2} \boldsymbol{\xi}'^2)} f(\mathbf{x}^j + \tau_j \boldsymbol{\xi}') \right) \\
&\leq c \varepsilon \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z} \frac{1}{|\mathbf{z}|^4} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z}' \frac{1}{|\mathbf{z}'|^4} \int d\boldsymbol{\xi} \langle \boldsymbol{\xi} \rangle^4 \sum_{\underline{m}, |\underline{m}| \leq 4} |D_{\boldsymbol{\xi}}^{\underline{m}} \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi})| \int d\boldsymbol{\xi}' \langle \boldsymbol{\xi}' \rangle^4 \sum_{\underline{n}, |\underline{n}| \leq 4} |D_{\boldsymbol{\xi}'}^{\underline{n}} \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi}')| \\
&\quad \cdot \sum_{\underline{k}, |\underline{k}| \leq 4} \sum_{\underline{l}, |\underline{l}| \leq 4} \int d\mathbf{x} \langle \mathbf{x}^j + \tau_j \boldsymbol{\xi} \rangle^4 |D_{\boldsymbol{\xi}}^{\underline{k}} f(\mathbf{x}^j + \tau_j \boldsymbol{\xi})| \langle \mathbf{x}^j + \tau_j \boldsymbol{\xi}' \rangle^4 |D_{\boldsymbol{\xi}'}^{\underline{l}} f(\mathbf{x}^j + \tau_j \boldsymbol{\xi}')| \quad (6.12)
\end{aligned}$$

Furthermore, in the last integral we apply the Schwartz inequality and we obtain

$$\sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,1}(t, \mathbf{x})|^2 \leq c \|\widetilde{V}\|_{W_4^{4,1}}^2 \left( \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} d\mathbf{z} \frac{1}{|\mathbf{z}|^4} \right)^2 \varepsilon \leq c \|\widetilde{V}\|_{W_4^{4,1}}^2 \varepsilon^3 \quad (6.13)$$

where the last integral in (6.13) has been explicitly computed.

The next step is to estimate the second term in the r.h.s of (6.9). We consider the Taylor expansion of the  $\varepsilon$ -dependent part of the integrand in (6.6) up to order one. We have

$$F_{\underline{n},j}^1(\boldsymbol{\xi}, L_\varepsilon \mathbf{z}; \mathbf{x}) e^{i\varepsilon(B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x}) + \mathbf{A}_{j,\varepsilon}^2(\mathbf{z}))} - F_{\underline{n},j}^1(\boldsymbol{\xi}, \mathbf{z}^c; \mathbf{x}) = g_{\underline{n},0}(\mathcal{R}_j^{-1} \boldsymbol{\xi}) \varepsilon \int_0^1 d\vartheta \frac{\partial T_j}{\partial \varepsilon}(\vartheta \varepsilon, \boldsymbol{\xi}, \mathbf{z}; \mathbf{x}) \quad (6.14)$$

where

$$T_j(\varepsilon, \boldsymbol{\xi}, \mathbf{z}; \mathbf{x}) \equiv \frac{e^{i\varepsilon(B_{j,\varepsilon}^2(\mathbf{z}; \mathbf{x}) + \mathbf{A}_{j,\varepsilon}^2(\mathbf{z}))}}{\sqrt{1 - \frac{\varepsilon^2}{v_0^2 \tau_j^2} (z_1^2 + z_2^2)}} e^{i(\boldsymbol{\xi} \cdot \mathbf{x}^j + \frac{\tau_j}{2} \boldsymbol{\xi}^2 + \varepsilon \frac{z_3}{2v_0} \boldsymbol{\xi}^2)} f(\mathbf{x}^j + \tau_j \boldsymbol{\xi} + \varepsilon \frac{z_3}{v_0} \boldsymbol{\xi}) \quad (6.15)$$

Substituting (6.14) into the second term in the r.h.s. of (6.9) and computing the sum over  $\underline{n}$  we obtain

$$\begin{aligned}
& \sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,2}(t, \mathbf{x})|^2 \\
&= \frac{\mathcal{N}_\varepsilon^2 \varepsilon^3}{v_0^6 \tau_j^4 (2\pi)^3} \int_{\Omega_\varepsilon} d\mathbf{z} \int_{\Omega_\varepsilon} d\mathbf{z}' \int d\mathbf{x} e^{-i\boldsymbol{\xi}^c \cdot (\mathbf{z} - \mathbf{z}')} \int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' e^{i\mathbf{z} \cdot \boldsymbol{\xi} - i\mathbf{z}' \cdot \boldsymbol{\xi}'} \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi}) \widetilde{V}(\mathcal{R}_j^{-1} \boldsymbol{\xi}') e^{-\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')^2}{4}} \\
&\quad \cdot \int_0^1 d\vartheta \frac{\partial T_j}{\partial \varepsilon}(\vartheta \varepsilon, \boldsymbol{\xi}, \mathbf{z}; \mathbf{x}) \int_0^1 d\vartheta' \frac{\partial T_j}{\partial \varepsilon}(\vartheta' \varepsilon, \boldsymbol{\xi}', \mathbf{z}'; \mathbf{x}) \quad (6.16)
\end{aligned}$$

We can now proceed with integration by parts along the same line of the previous case and we find the same kind of result

$$\sum_{\underline{n}} \int d\mathbf{x} |Q_{j,\underline{n}}^{\varepsilon,2}(t, \mathbf{x})|^2 \leq c \|\tilde{V}\|_{W_4^{4,1}}^2 \varepsilon^3 \quad (6.17)$$

Using the estimates (6.13), (6.17) in (6.9) we obtain the estimate (3.15) and therefore we also conclude the proof of proposition 3.1.  $\square$

## 7. NON STATIONARY TERMS

For the estimate of the non stationary terms it will be useful the following technical lemma.

**Lemma 7.1.** *For any  $s_1, \dots, s_{l-1} > 0$ ,  $l > 1$ , let  $\zeta_{s_1, \dots, s_{l-1}} : \mathbb{R}^{3l} \rightarrow \mathbb{C}$  be defined by*

$$\zeta_{s_1, \dots, s_{l-1}}(\xi_1, \dots, \xi_l) \equiv \langle \phi_{\underline{0}}, e^{i\xi_l \cdot (\cdot)} U(s_{l-1}) e^{-i\xi_{l-1} \cdot (\cdot)} U(s_{l-2}) \dots e^{-i\xi_2 \cdot (\cdot)} U(s_1) e^{-i\xi_1 \cdot (\cdot)} \phi_{\underline{0}} \rangle \quad (7.1)$$

*Then  $\zeta_{s_1, \dots, s_{l-1}} \in C^\infty(\mathbb{R}^{3l})$  and for every  $\underline{\alpha}_1, \dots, \underline{\alpha}_l \in \mathbb{N}^3$  there exists  $c_{\underline{\alpha}_1, \dots, \underline{\alpha}_l}$ , independent of  $s_1, \dots, s_{l-1}$ , such that the following estimate holds*

$$\|D_{\xi_1}^{\underline{\alpha}_1} \dots D_{\xi_l}^{\underline{\alpha}_l} \zeta_{s_1, \dots, s_{l-1}}(\xi_1, \dots, \xi_l)\| \leq c_{\underline{\alpha}_1, \dots, \underline{\alpha}_l} \left( \sum_{i=1}^l \langle \xi_i \rangle \right)^{|\alpha_1| + \dots + |\alpha_l|} \quad (7.2)$$

The proof is a direct generalization of the proof of the one dimensional case given in lemma 3.1 in [FinT] and it is omitted. Moreover by formula (6.2) we have

$$\sum_{\underline{n}} e^{-i\frac{|\underline{n}|}{\varepsilon}t} g_{\underline{n},\underline{0}}(\xi) \bar{g}_{\underline{n},\underline{0}}(\xi) = \frac{e^{i\frac{3}{2\varepsilon}t}}{(2\pi)^3} \tilde{V}(\xi) \overline{\tilde{V}}(\xi') \zeta_{t/\varepsilon}(\xi, \xi') \quad (7.3)$$

Using these facts, in the following we prove proposition 3.2.

From the explicit expression of  $\mathcal{T}_{j,\underline{n}}^\varepsilon(t, \mathbf{x})$  given in proposition 4.1 we have

$$\begin{aligned} & \|I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)\|^2 \\ &= \varepsilon^3 \sum_{\underline{n}} \int d\mathbf{x} |\mathcal{T}_{j,\underline{n}}^\varepsilon(t, \mathbf{x})|^2 = \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}} \int d\mathbf{x} \int_0^t ds \int d\xi \int_{S^2 \setminus \mathcal{C}_j} d\hat{\mathbf{u}} F_{\underline{n}}(\xi, s; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_{\underline{n},j}(\xi, s, \hat{\mathbf{u}}; \mathbf{x})} \\ & \cdot \int_0^t ds' \int d\xi' \int_{S^2 \setminus \mathcal{C}_j} d\hat{\mathbf{u}}' \overline{F_{\underline{n}}}(\xi', s'; \mathbf{x}) e^{-\frac{i}{\varepsilon} \Phi_{\underline{n},j}(\xi', s', \hat{\mathbf{u}}'; \mathbf{x})} \end{aligned} \quad (7.4)$$

Taking into account the expression of  $F_{\underline{n}}$  and  $\Phi_{\underline{n},j}$  (see (4.4), (4.5), (4.6)) and formula (6.2), we compute the sum over  $\underline{n}$  in (7.4)

$$\begin{aligned} & \|I_j^\varepsilon(t) (\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)\|^2 = \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2 (2\pi)^3} \int d\mathbf{x} \int_0^t ds \int_0^t ds' e^{i\frac{3}{2\varepsilon}(s'-s)} \int_{S^2 \setminus \mathcal{C}_j} d\hat{\mathbf{u}} \int_{S^2 \setminus \mathcal{C}_j} d\hat{\mathbf{u}}' e^{\frac{i}{\varepsilon} v_0(\hat{\mathbf{u}} - \hat{\mathbf{u}}') \cdot \mathbf{x}} \\ & \cdot \int d\xi' \int d\xi e^{-\frac{i}{\varepsilon} \Phi_j(\xi', s', \hat{\mathbf{u}}')} e^{\frac{i}{\varepsilon} \Phi_j(\xi, s, \hat{\mathbf{u}})} G_\varepsilon(\xi, \xi', s, s'; \mathbf{x}) \end{aligned} \quad (7.5)$$

where

$$G_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', s, s'; \mathbf{x}) \equiv \tilde{V}(\boldsymbol{\xi}) \overline{\tilde{V}}(\boldsymbol{\xi}') e^{i(\mathbf{x} \cdot \boldsymbol{\xi} + \frac{s}{2} \xi^2)} f(\mathbf{x} + s \boldsymbol{\xi}) e^{-i(\mathbf{x} \cdot \boldsymbol{\xi}' + \frac{s'}{2} \xi'^2)} f(\mathbf{x} + s' \boldsymbol{\xi}') \zeta_{(s-s')/\varepsilon}(\boldsymbol{\xi}, \boldsymbol{\xi}') \quad (7.6)$$

$$\Phi_j(\boldsymbol{\xi}, s, \hat{\mathbf{u}}) \equiv v_0 \boldsymbol{\xi} \cdot (s \hat{\mathbf{u}} - \tau_j \hat{\mathbf{a}}_j) \quad (7.7)$$

Let us consider the last two oscillatory integrals in the variables  $\boldsymbol{\xi}, \boldsymbol{\xi}'$  in formula (7.5). We notice that

$$|\nabla_{\boldsymbol{\xi}} \Phi_j|^2 = v_0^2 (s^2 + \tau_j^2 - 2s\tau_j \hat{\mathbf{u}} \cdot \hat{\mathbf{a}}_j) = v_0^2 (s^2 + \tau_j^2 - 2s\tau_j (\mathcal{R}_j \hat{\mathbf{u}})_3) \quad (7.8)$$

For  $\hat{\mathbf{u}} \in S^2 \setminus \mathcal{C}_j$  we have  $(\mathcal{R}_j \hat{\mathbf{u}})_3 = \sqrt{1 - (\mathcal{R}_j \hat{\mathbf{u}})_1^2 - (\mathcal{R}_j \hat{\mathbf{u}})_2^2} \leq \cos \theta_0$  and therefore

$$\begin{aligned} |\nabla_{\boldsymbol{\xi}} \Phi_j|^2 &\geq v_0^2 (s^2 + \tau_j^2 - 2s\tau_j \cos \theta_0) \geq v_0^2 \min_s (s^2 + \tau_j^2 - 2s\tau_j \cos \theta_0) \\ &= v_0^2 \tau_j^2 \sin^2 \theta_0 \geq v_0^2 \tau_1^2 \sin^2 \theta_0 \equiv \Delta^2 \end{aligned} \quad (7.9)$$

Using the above inequality, we can estimate the two integrals in the variables  $\boldsymbol{\xi}, \boldsymbol{\xi}'$  in (7.5) exploiting a non-stationary phase argument. With a repeated application  $k$  times of the identity

$$a e^{ib} = -i \operatorname{div} \left( e^{ib} \frac{\nabla b}{|\nabla b|^2} a \right) + i e^{ib} \operatorname{div} \left( \frac{\nabla b}{|\nabla b|^2} a \right) \quad (7.10)$$

we have

$$\begin{aligned} &\int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' e^{\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}, s, \hat{\mathbf{u}}; \mathbf{x})} e^{-\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}', s', \hat{\mathbf{u}}'; \mathbf{x})} G_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', s, s'; \mathbf{x}) = \\ &\varepsilon^{2k} \int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' e^{\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}, s, \hat{\mathbf{u}}; \mathbf{x})} e^{-\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}', s', \hat{\mathbf{u}}'; \mathbf{x})} L_{\boldsymbol{\xi}}^k L_{\boldsymbol{\xi}'}^k G_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', s, s'; \mathbf{x}) \end{aligned} \quad (7.11)$$

where

$$L_{\boldsymbol{\xi}} G_\varepsilon = \sum_{j=1}^3 u_j \frac{\partial G_\varepsilon}{\partial \xi_j}, \quad u_j = \frac{1}{|\nabla_{\boldsymbol{\xi}} \Theta|^2} \frac{\partial \Theta}{\partial \xi_j} \quad (7.12)$$

and moreover

$$L_{\boldsymbol{\xi}}^k G_\varepsilon = \sum_{\underline{m}, |\underline{m}|=k} u_1^{m_1} u_2^{m_2} u_3^{m_3} D_{\boldsymbol{\xi}}^{\underline{m}} G_\varepsilon \quad (7.13)$$

We notice that

$$|u_j| \leq \frac{1}{|\nabla_{\boldsymbol{\xi}} \Theta|} < \frac{1}{\Delta} \quad (7.14)$$

Therefore

$$\begin{aligned} &\left| \int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' G_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', s, s'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}, s, \hat{\mathbf{u}}; \mathbf{x})} e^{-\frac{i}{\varepsilon} \Phi_j(\boldsymbol{\xi}', s', \hat{\mathbf{u}}'; \mathbf{x})} \right| \\ &\leq \frac{\varepsilon^{2k}}{\Delta^{2k}} \sum_{\underline{m}, |\underline{m}|=k} \sum_{\underline{m}', |\underline{m}'|=k} \int d\boldsymbol{\xi} \int d\boldsymbol{\xi}' \left| D_{\boldsymbol{\xi}'}^{\underline{m}'} D_{\boldsymbol{\xi}}^{\underline{m}} G_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', s, s'; \mathbf{x}) \right| \end{aligned} \quad (7.15)$$

The next step is to estimate the derivatives of  $G_\varepsilon$ . From (7.6) we have

$$\begin{aligned}
& \sum_{\underline{m}, |\underline{m}|=k} \sum_{\underline{m}', |\underline{m}'|=k} \left| D_{\xi'}^{\underline{m}'} D_{\xi}^{\underline{m}} G_\varepsilon(\xi, \xi', s, s'; \mathbf{x}) \right| \\
& \leq c_k \sum_{\underline{n}, |\underline{n}| \leq k} |D_{\xi}^{\underline{n}} \zeta_{(s-s')/\varepsilon}(\xi, \xi')| \sum_{\underline{l}, |\underline{l}| \leq k} |D_{\xi}^{\underline{l}} \tilde{V}(\xi)| \sum_{\underline{l}', |\underline{l}'| \leq k} |D_{\xi'}^{\underline{l}'} \tilde{V}(\xi')| \langle \mathbf{x} + s\xi \rangle^k \langle \mathbf{x} + s'\xi' \rangle^k \\
& \quad \langle s \rangle^k \langle s' \rangle^k \sum_{\underline{p}, |\underline{p}| \leq k} |D_{\mathbf{x}}^{\underline{p}} f(\mathbf{x} + s\xi)| \sum_{\underline{p}', |\underline{p}'| \leq k} |D_{\mathbf{x}}^{\underline{p}'} f(\mathbf{x} + s'\xi')| \\
& \leq c_k \langle \xi \rangle^k \langle \xi' \rangle^k \sum_{\underline{l}, |\underline{l}| \leq k} |D_{\xi}^{\underline{l}} \tilde{V}(\xi)| \sum_{\underline{l}', |\underline{l}'| \leq k} |D_{\xi'}^{\underline{l}'} \tilde{V}(\xi')| \langle \mathbf{x} + s\xi \rangle^k \langle \mathbf{x} + s'\xi' \rangle^k \\
& \quad \langle s \rangle^k \langle s' \rangle^k \sum_{\underline{p}, |\underline{p}| \leq k} |D_{\mathbf{x}}^{\underline{p}} f(\mathbf{x} + s\xi)| \sum_{\underline{p}', |\underline{p}'| \leq k} |D_{\mathbf{x}}^{\underline{p}'} f(\mathbf{x} + s'\xi')| \tag{7.16}
\end{aligned}$$

where, in the last step, we have used lemma 7.1. Therefore

$$\begin{aligned}
& \left| \int d\xi \int d\xi' G_\varepsilon(\xi, \xi', s, s'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Phi_j(\xi, s, \hat{\mathbf{u}})} e^{-\frac{i}{\varepsilon} \Phi_j(\xi', s', \hat{\mathbf{u}}')} \right| \\
& \leq c_k \frac{\varepsilon^{2k}}{\Delta^{2k}} \langle s \rangle^k \langle s' \rangle^k \int d\xi \int d\xi' \langle \xi \rangle^k \langle \xi' \rangle^k \sum_{\underline{l}, |\underline{l}| \leq k} |D_{\xi}^{\underline{l}} \tilde{V}(\xi)| \sum_{\underline{l}', |\underline{l}'| \leq k} |D_{\xi'}^{\underline{l}'} \tilde{V}(\xi')| \\
& \quad \langle \mathbf{x} + s\xi \rangle^k \langle \mathbf{x} + s'\xi' \rangle^k \sum_{\underline{p}, |\underline{p}| \leq k} |D_{\mathbf{x}}^{\underline{p}} f(\mathbf{x} + s\xi)| \sum_{\underline{p}', |\underline{p}'| \leq k} |D_{\mathbf{x}}^{\underline{p}'} f(\mathbf{x} + s'\xi')| \tag{7.17}
\end{aligned}$$

Using (7.17) in (7.5) we have

$$\begin{aligned}
& \|I_j^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)\|^2 \\
& \leq c_k \varepsilon^{2k-2} t^{2k+2} \int d\xi \int d\xi' \langle \xi \rangle^k \langle \xi' \rangle^k \sum_{\underline{l}, |\underline{l}| \leq k} |D_{\xi}^{\underline{l}} \tilde{V}(\xi)| \sum_{\underline{l}', |\underline{l}'| \leq k} |D_{\xi'}^{\underline{l}'} \tilde{V}(\xi')| \\
& \quad \int d\mathbf{x} \langle \mathbf{x} + s\xi \rangle^k \langle \mathbf{x} + s'\xi' \rangle^k \sum_{\underline{p}, |\underline{p}| \leq k} |D_{\mathbf{x}}^{\underline{p}} f(\mathbf{x} + s\xi)| \sum_{\underline{p}', |\underline{p}'| \leq k} |D_{\mathbf{x}}^{\underline{p}'} f(\mathbf{x} + s'\xi')| \tag{7.18}
\end{aligned}$$

Finally we use the Schwartz inequality in the last integral and we obtain

$$\|I_j^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,j}^\varepsilon)\|^2 \leq c_k t^{2k+2} \|\tilde{V}\|_{W_k^{k,1}}^2 \varepsilon^{2k-2} \tag{7.19}$$

concluding the proof of the proposition 3.2.

## 8. ESTIMATE OF THE REST

In this section we prove the proposition 3.3. From (4.15) we have

$$\|J^\varepsilon(t)\Psi_0^\varepsilon\| \leq \sum_{\substack{k,l=1 \\ k \neq l}}^N \|J_{k,l}^\varepsilon(t)\Psi_0^\varepsilon\| + \sum_{k=1}^N \|J_k^\varepsilon(t)\Psi_0^\varepsilon\| \quad (8.1)$$

Let us estimate the first term in the r.h.s of (8.1). Using (4.18), (4.20), (4.22) and (4.24) we obtain

$$\begin{aligned} & \|J_{k,l}^\varepsilon(t)\Psi_0^\varepsilon\|^2 \\ &= \varepsilon^3 \sum_{\underline{n}, \underline{m}} \int d\mathbf{x} \left| \mathcal{J}_{\underline{n}, \underline{m}}^{\varepsilon, k, l}(t, \mathbf{x}) \right|^2 \\ &= \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}, \underline{m}} \int d\mathbf{x} \left| \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{S^2} d\hat{\mathbf{u}} G_{\underline{n}, \underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^{k, l}(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x})} \right|^2 \\ &= \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}, \underline{m}} \int d\mathbf{x} \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{S^2} d\hat{\mathbf{u}} \int_0^t ds' \int_0^{s'} d\sigma' \int d\xi' \int d\eta' \int_{S^2} d\hat{\mathbf{u}}' \\ & \quad e^{i(\eta+\xi) \cdot \mathbf{x} + i(\frac{s}{2}\eta'^2 + \frac{\sigma}{2}\xi'^2) + is\eta \cdot \xi} f(\mathbf{x} + \sigma \xi + s \eta) e^{-\frac{i}{\varepsilon} [\xi \cdot \mathbf{a}_l + \eta \cdot \mathbf{a}_k - v_0 \hat{\mathbf{u}} \cdot (\mathbf{x} + \sigma \xi + s \eta)]} \\ & \quad e^{-i(\eta' + \xi') \cdot \mathbf{x} - i(\frac{s'}{2}\eta'^2 + \frac{\sigma'}{2}\xi'^2) - is'\eta' \cdot \xi'} f(\mathbf{x} + \sigma' \xi' + s' \eta') e^{\frac{i}{\varepsilon} [\xi' \cdot \mathbf{a}_l + \eta' \cdot \mathbf{a}_k - v_0 \hat{\mathbf{u}}' \cdot (\mathbf{x} + \sigma' \xi' + s' \eta')]} \\ & \quad \sum_{\underline{n}} e^{i|\underline{n}| \frac{s-s'}{\varepsilon}} g_{\underline{n}, \underline{0}}(\eta) \bar{g}_{\underline{n}, \underline{0}}(\eta') \sum_{\underline{m}} e^{i|\underline{m}| \frac{\sigma-\sigma'}{\varepsilon}} g_{\underline{m}, \underline{0}}(\xi) \bar{g}_{\underline{m}, \underline{0}}(\xi') \end{aligned} \quad (8.2)$$

Taking into account formula (6.2) we compute the sum over  $\underline{n}, \underline{m}$

$$\begin{aligned} & \|J_{k,l}^\varepsilon(t)\Psi_0^\varepsilon\|^2 \\ &= \frac{\mathcal{N}_\varepsilon^2}{(2\pi)^6 \varepsilon^2} \int d\mathbf{x} \int_0^t ds \int_0^t ds' \int_0^s d\sigma \int_0^{s'} d\sigma' e^{i\frac{3}{2\varepsilon}(s'-s)} e^{i\frac{3}{2\varepsilon}(\sigma'-\sigma)} \int_{S^2} d\hat{\mathbf{u}} \int_{S^2} d\hat{\mathbf{u}}' e^{-\frac{i}{\varepsilon}(\hat{\mathbf{u}} - \hat{\mathbf{u}}') \cdot \mathbf{x}} \\ & \quad \int d\xi \int d\xi' \int d\eta \int d\eta' A^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{kl}(\xi, \eta, s, \sigma, \hat{\mathbf{u}})} e^{-\frac{i}{\varepsilon} \Theta_{kl}(\xi', \eta', s', \sigma', \hat{\mathbf{u}})} \end{aligned} \quad (8.3)$$

where

$$\begin{aligned} A^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) &= e^{i(\eta+\xi) \cdot \mathbf{x} + i(\frac{s}{2}\eta'^2 + \frac{\sigma}{2}\xi'^2) + is\eta \cdot \xi} e^{-i(\eta' + \xi') \cdot \mathbf{x} - i(\frac{s'}{2}\eta'^2 + \frac{\sigma'}{2}\xi'^2) - is'\eta' \cdot \xi'} \\ & \quad \widetilde{V}(\eta) \widetilde{V}(\eta') \widetilde{V}(\xi) \widetilde{V}(\xi') \zeta_{(s'-s)/\varepsilon}(\eta, \eta') \zeta_{(\sigma'-\sigma)/\varepsilon}(\xi, \xi') f(\mathbf{x} + \sigma \xi + s \eta) f(\mathbf{x} + \sigma' \xi' + s' \eta') \end{aligned} \quad (8.4)$$

$$\Theta_{kl}(\xi, \eta, s, \sigma, \hat{\mathbf{u}}) = v_0 \xi \cdot (\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l) + v_0 \eta \cdot (s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k) \quad (8.5)$$



Let us consider the oscillatory integrals in the variables  $\xi, \xi', \eta, \eta'$  in formula (8.3). We observe that the  $|\nabla_{\xi, \eta} \Theta_{kl}|$  doesn't vanish for  $\hat{\mathbf{u}} \in S^2$ . In fact one has

$$\begin{aligned}
|\nabla_{\xi, \eta} \Theta_{kl}|^2 &= (v_0^2 |\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l|^2 + v_0^2 |s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k|^2) \geq \min_{\hat{\mathbf{u}} \in S^2} (v_0^2 |\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l|^2 + v_0^2 |s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k|^2) \\
&= \min \left\{ \min_{\hat{\mathbf{u}} \in S^2 \setminus \mathcal{C}_k} (v_0^2 |\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l|^2 + v_0^2 |s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k|^2), \min_{\hat{\mathbf{u}} \in \mathcal{C}_k} (v_0^2 |\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l|^2 + v_0^2 |s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k|^2) \right\} \\
&\geq \min \left\{ \min_{\hat{\mathbf{u}} \in S^2 \setminus \mathcal{C}_k} (v_0^2 |s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k|^2), \min_{\hat{\mathbf{u}} \in \mathcal{C}_k} (v_0^2 |\sigma \hat{\mathbf{u}} - \tau_l \hat{\mathbf{a}}_l|^2) \right\} \\
&\geq \min \left\{ v_0^2 \tau_k^2 \sin^2 \theta_0, v_0^2 \tau_l^2 \sin^2 \theta_0 \right\} \geq v_0^2 \tau_1^2 \sin^2 \theta_0 = \Delta^2
\end{aligned} \tag{8.6}$$

where in the last step we have used (7.9). Hence we can estimate the integrals in the variables  $\xi, \xi', \eta, \eta'$  exploiting a non-stationary phase argument. With a repeated application of the identity (7.10) and using (8.6) we have

$$\begin{aligned}
&\left| \int d\xi \int d\xi' \int d\eta \int d\eta' A^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{kl}(\xi, \eta, s, \sigma, \hat{\mathbf{u}})} e^{-\frac{i}{\varepsilon} \Theta_{kl}(\xi', \eta', s', \sigma', \hat{\mathbf{u}})} \right| \\
&\leq \frac{\varepsilon^{2d}}{\Delta^{2d}} \sum_{\substack{\underline{m}_1, \underline{m}_2, \underline{m}'_1, \underline{m}'_2 \\ |\underline{m}_1| + |\underline{m}_2| = d \\ |\underline{m}'_1| + |\underline{m}'_2| = d}} \int d\xi \int d\xi' \int d\eta \int d\eta' \left| D_{\xi}^{\underline{m}_1} D_{\eta}^{\underline{m}_2} D_{\xi'}^{\underline{m}'_1} D_{\eta'}^{\underline{m}'_2} A^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) \right|
\end{aligned} \tag{8.7}$$

for any integer  $d > 0$ . The next step is to estimate the derivatives of  $A^\varepsilon$ . Proceeding as in the previous section we obtain

$$\begin{aligned}
&\left| \int d\xi \int d\xi' \int d\eta \int d\eta' A^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{kl}(\xi, \eta, s, \sigma, \hat{\mathbf{u}})} e^{-\frac{i}{\varepsilon} \Theta_{kl}(\xi', \eta', s', \sigma', \hat{\mathbf{u}})} \right| \\
&\leq c_d \frac{\varepsilon^{2d}}{\Delta^{2d}} \langle s \rangle^d \langle s' \rangle^d \langle \sigma \rangle^d \langle \sigma' \rangle^d \int d\xi \int d\xi' \langle \xi \rangle^d \langle \xi' \rangle^d \sum_{\underline{l}, |\underline{l}| \leq d} |D_{\xi}^{\underline{l}} \widetilde{V}(\xi)| \sum_{\underline{l}', |\underline{l}'| \leq d} |D_{\xi'}^{\underline{l}'} \widetilde{V}(\xi')| \\
&\int d\eta \int d\eta' \langle \eta \rangle^d \langle \eta' \rangle^d \sum_{\underline{p}, |\underline{p}| \leq d} |D_{\eta}^{\underline{p}} \widetilde{V}(\eta)| \sum_{\underline{p}', |\underline{p}'| \leq d} |D_{\eta'}^{\underline{p}'} \widetilde{V}(\eta')| \\
&\langle \mathbf{x} + \sigma \xi + s \eta \rangle^d \langle \mathbf{x} + \sigma' \xi' + s' \eta' \rangle^d \sum_{\underline{q}, |\underline{q}| \leq d} |D_{\mathbf{x}}^{\underline{q}} f(\mathbf{x} + \sigma \xi + s \eta)| \sum_{\underline{q}', |\underline{q}'| \leq d} |D_{\mathbf{x}}^{\underline{q}'} f(\mathbf{x} + \sigma' \xi' + s' \eta')|
\end{aligned} \tag{8.8}$$

Then, substituting (8.8) in (8.3) and using the Schwartz inequality in the integral with respect to the variable  $\mathbf{x}$ , we have the following estimate of the first term in the r.h.s. of (8.1)

$$\|J_{k,l}^\varepsilon(t) \Psi_0^\varepsilon\|^2 \leq c_d t^{4d+4} \|\widetilde{V}\|_{W_d^{d,1}}^4 \varepsilon^{2d-2} \tag{8.9}$$

Let us analyze the second term in the r.h.s of (8.1). We write

$$\|J_k^\varepsilon(t)\Psi_0^\varepsilon\| \leq \|J_k^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,k}^\varepsilon)\| + \|J_k^\varepsilon(t)\Psi_{0,k}^\varepsilon\| \quad (8.10)$$

From definition (3.12) and from (4.17) we have

$$\begin{aligned} & \|J_k^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,k}^\varepsilon)\|^2 = \\ & \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}} \left| \sum_{\underline{m}} \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{S^2 \setminus \mathcal{C}_k} d\hat{\mathbf{u}} L_{\underline{n},\underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n},\underline{m}}^k(\xi, \eta, s, \sigma; \hat{\mathbf{u}}; \mathbf{x})} \right|^2 \end{aligned} \quad (8.11)$$

and  $L_{\underline{n},\underline{m}}, \Theta_{\underline{n},\underline{m}}^k$  are defined by (4.21), (4.23). The sum over  $\underline{m}$  in (8.11) can be computed as follows

$$\begin{aligned} & \sum_{\underline{m}} e^{\frac{i}{\varepsilon} |m|(\sigma-s)} g_{\underline{n},\underline{m}}(\eta) g_{\underline{m},\underline{0}}(\xi) \\ &= \frac{\tilde{V}(\xi) \tilde{V}(\eta)}{(2\pi)^3} \sum_{\underline{m}} \int d\mathbf{y} \phi_{\underline{n}}(\mathbf{y}) \phi_{\underline{m}}(\mathbf{y}) e^{-i\mathbf{y} \cdot \eta} \int d\mathbf{y}' \phi_{\underline{0}}(\mathbf{y}') e^{i\mathbf{y}' \cdot \xi} e^{\frac{i}{\varepsilon} |m|(\sigma-s)} \\ &= \frac{\tilde{V}(\xi) \tilde{V}(\eta)}{(2\pi)^3} \int d\mathbf{y} \overline{\phi_{\underline{n}}(\mathbf{y}) e^{-i\mathbf{y} \cdot \eta}} \int d\mathbf{y}' \sum_{\underline{m}} \phi_{\underline{m}}(\mathbf{y}) \phi_{\underline{m}}(\mathbf{y}') e^{i\frac{|m|}{\varepsilon}(\sigma-s)} \phi_{\underline{0}}(\mathbf{y}') e^{-i\mathbf{y}' \cdot \xi} \\ &= \frac{\tilde{V}(\xi) \tilde{V}(\eta)}{(2\pi)^3} e^{i\frac{3}{2\varepsilon}(\sigma-s)} \langle \phi_{\underline{n}}, e^{i\eta \cdot (\cdot)} U\left(\frac{s-\sigma}{\varepsilon}\right) e^{-i\xi \cdot (\cdot)} \phi_{\underline{0}} \rangle \end{aligned} \quad (8.12)$$

therefore

$$\begin{aligned} & \sum_{\underline{n}} e^{\frac{i}{\varepsilon} |n|(s-s')} \sum_{\underline{m}} e^{\frac{i}{\varepsilon} |m|(\sigma-s)} g_{\underline{n},\underline{m}}(\eta) g_{\underline{m},\underline{0}}(\xi) \sum_{\underline{m}'} e^{-\frac{i}{\varepsilon} |m'|(\sigma'-s')} \bar{g}_{\underline{n},\underline{m}'}(\eta') \bar{g}_{\underline{m}',\underline{0}}(\xi') \\ &= \frac{\tilde{V}(\xi) \tilde{V}(\eta) \tilde{V}(\xi') \tilde{V}(\eta') e^{i\frac{3}{2\varepsilon}(\sigma-\sigma')}}{(2\pi)^6} \\ & \cdot \langle e^{i\eta' \cdot (\cdot)} U\left(\frac{s'-\sigma'}{\varepsilon}\right) e^{-i\xi' \cdot (\cdot)} \phi_{\underline{0}}, U\left(\frac{s'-s}{\varepsilon}\right) e^{i\eta \cdot (\cdot)} U\left(\frac{s-\sigma}{\varepsilon}\right) e^{-i\xi \cdot (\cdot)} \phi_{\underline{0}} \rangle \\ &= \frac{\tilde{V}(\xi) \tilde{V}(\eta) \tilde{V}(\xi') \tilde{V}(\eta') e^{i\frac{3}{2\varepsilon}(\sigma-\sigma')}}{(2\pi)^6} \zeta_{s_1, s_2, s_3}(\xi', \eta', \eta, \xi) \end{aligned} \quad (8.13)$$

where  $s_1 = (\sigma' - s')/\varepsilon$ ,  $s_2 = (s' - s)/\varepsilon$ ,  $s_3 = (s - \sigma)/\varepsilon$ . Using (8.13) we have

$$\begin{aligned} & \|J_k^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,k}^\varepsilon)\|^2 = \\ & \frac{\mathcal{N}_\varepsilon^2}{(2\pi)^6 \varepsilon^2} \int d\mathbf{x} \int_0^t ds \int_0^s d\sigma \int_0^t ds' \int_0^s d\sigma' e^{\frac{3i}{2\varepsilon}(\sigma-\sigma')} \int_{S^2 \setminus \mathcal{C}_k} d\hat{\mathbf{u}} \int_{S^2 \setminus \mathcal{C}_k} d\hat{\mathbf{u}}' e^{\frac{i}{\varepsilon} v_0(\hat{\mathbf{u}} - \hat{\mathbf{u}}') \cdot \mathbf{x}} \\ & \cdot \int d\xi \int d\eta \int d\xi' \int d\eta' L^\varepsilon(\xi, \xi', \eta, \eta', s, s', \sigma, \sigma'; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_k(\xi, \eta, s, \sigma; \hat{\mathbf{u}})} e^{-\frac{i}{\varepsilon} \Theta_k(\xi', \eta', s', \sigma'; \hat{\mathbf{u}}')} \end{aligned} \quad (8.14)$$

where

$$\begin{aligned}
L^\varepsilon(\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\eta}, \boldsymbol{\eta}', s, s', \sigma, \sigma'; \mathbf{x}) &= e^{i(\boldsymbol{\eta}+\boldsymbol{\xi}) \cdot \mathbf{x} + i(\frac{s}{2}\boldsymbol{\eta}^2 + \frac{\sigma}{2}\boldsymbol{\xi}^2) + is\boldsymbol{\eta} \cdot \boldsymbol{\xi}} f(\mathbf{x} + \sigma \boldsymbol{\xi} + s \boldsymbol{\eta}) \\
&\cdot e^{-i(\boldsymbol{\eta}'+\boldsymbol{\xi}') \cdot \mathbf{x} - i(\frac{s'}{2}\boldsymbol{\eta}'^2 + \frac{\sigma'}{2}\boldsymbol{\xi}'^2) - is'\boldsymbol{\eta}' \cdot \boldsymbol{\xi}'} f(\mathbf{x} + \sigma' \boldsymbol{\xi}' + s' \boldsymbol{\eta}') \widetilde{V}(\boldsymbol{\xi}) \widetilde{V}(\boldsymbol{\xi}') \widetilde{V}(\boldsymbol{\eta}) \widetilde{V}(\boldsymbol{\eta}') \zeta_{s_1, s_2, s_3}(\boldsymbol{\xi}', \boldsymbol{\eta}', \boldsymbol{\eta}, \boldsymbol{\xi}) \\
\Theta_k(\boldsymbol{\xi}, \boldsymbol{\eta}, s, \sigma, \hat{\mathbf{u}}; \mathbf{x}) &= v_0 (\sigma \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k) \cdot \boldsymbol{\xi} + v_0 (s \hat{\mathbf{u}} - \tau_k \hat{\mathbf{a}}_k) \cdot \boldsymbol{\eta}
\end{aligned} \tag{8.15}$$

We observe that the  $|\nabla_{\boldsymbol{\xi}, \boldsymbol{\eta}} \Theta_k|$  doesn't vanish in  $S^2 \setminus \mathcal{C}_k$ . In fact we have

$$|\nabla_{\boldsymbol{\xi}, \boldsymbol{\eta}} \Theta_k| \geq v_0 \tau_k \sin \theta_0 \geq v_0 \tau_1 \sin \theta_0 \equiv \Delta \tag{8.16}$$

Hence we can estimate the integrals in the variables  $\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\eta}, \boldsymbol{\eta}'$  exploiting a non-stationary phase argument along the same line of the previous case (see (8.7), (8.8)) and we find

$$\|J_k^\varepsilon(t)(\Psi_0^\varepsilon - \Psi_{0,k}^\varepsilon)\|^2 \leq c_d t^{4d+4} \|\tilde{V}\|_{W_d^{d,1}}^4 \varepsilon^{2d-2} \tag{8.17}$$

for any integer  $d > 0$ .

Finally we consider the last term in (8.10). We have

$$\|J_k^\varepsilon(t)\Psi_{0,k}^\varepsilon\|^2 = \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}} \left| \sum_{\underline{m}} \int_0^t ds \int_0^s d\sigma \int d\boldsymbol{\xi} \int d\boldsymbol{\eta} \int_{\mathcal{C}_k} d\hat{\mathbf{u}} L_{\underline{n}, \underline{m}}(\boldsymbol{\xi}, \boldsymbol{\eta}, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^k(\boldsymbol{\xi}, \boldsymbol{\eta}, s, \sigma \hat{\mathbf{u}}; \mathbf{x})} \right|^2 \tag{8.18}$$

We rewrite the integral in the variables  $s, \sigma, \boldsymbol{\xi}, \boldsymbol{\eta}$  in a more convenient form. In particular by the change of variables  $\boldsymbol{\xi} \rightarrow \mathcal{R}_k \boldsymbol{\xi}$ ,  $\boldsymbol{\eta} \rightarrow \mathcal{R}_k \boldsymbol{\eta}$ ,  $\hat{\mathbf{u}} \rightarrow \mathcal{R}_k \hat{\mathbf{u}}$  with  $\mathcal{R}_k$  defined in (3.7), we obtain

$$\begin{aligned}
&\int_0^t ds \int_0^s d\sigma \int d\boldsymbol{\xi} \int d\boldsymbol{\eta} \int_{\mathcal{C}_k} d\hat{\mathbf{u}} L_{\underline{n}, \underline{m}}(\boldsymbol{\xi}, \boldsymbol{\eta}, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^k(\boldsymbol{\xi}, \boldsymbol{\eta}, s, \sigma \hat{\mathbf{u}}; \mathbf{x})} \\
&= \int_0^t ds \int_0^s d\sigma \int d\boldsymbol{\xi} \int d\boldsymbol{\eta} \int_{\mathcal{C}_0} d\hat{\mathbf{u}} L_{\underline{n}, \underline{m}}(\mathcal{R}_k^{-1} \boldsymbol{\xi}, \mathcal{R}_k^{-1} \boldsymbol{\eta}, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^k(\mathcal{R}_k^{-1} \boldsymbol{\xi}, \mathcal{R}_k^{-1} \boldsymbol{\eta}, s, \sigma \hat{\mathbf{u}}; \mathbf{x})}
\end{aligned} \tag{8.19}$$

where

$$\begin{aligned}
&L_{\underline{n}, \underline{m}}(\mathcal{R}_k^{-1} \boldsymbol{\xi}, \mathcal{R}_k^{-1} \boldsymbol{\eta}, s, \sigma; \mathbf{x}) \\
&= e^{i(\boldsymbol{\eta}+\boldsymbol{\xi}) \cdot \mathbf{x}^k + i(\frac{s}{2}\boldsymbol{\eta}^2 + \frac{\sigma}{2}\boldsymbol{\xi}^2) + is\boldsymbol{\eta} \cdot \boldsymbol{\xi}} g_{\underline{n}, \underline{m}}(\mathcal{R}_k^{-1} \boldsymbol{\eta}) g_{\underline{m}, \underline{0}}(\mathcal{R}_k^{-1} \boldsymbol{\xi}) f(\mathbf{x}^k + \sigma \boldsymbol{\xi} + s \boldsymbol{\eta})
\end{aligned} \tag{8.20}$$

$$\begin{aligned}
&\Theta_{\underline{n}, \underline{m}}^k(\mathcal{R}_k^{-1} \boldsymbol{\xi}, \mathcal{R}_k^{-1} \boldsymbol{\eta}, s, \sigma, \hat{\mathbf{u}}; \mathbf{x}) \\
&= -(\xi_3 + \eta_3) |\mathbf{a}_k| + v_0 \hat{\mathbf{u}} \cdot (\mathbf{x}^k + \sigma \boldsymbol{\xi} + s \boldsymbol{\eta}) + (|n| - |m|)s + |m|\sigma
\end{aligned} \tag{8.21}$$

and  $\mathbf{x}^k \equiv \mathcal{R}_k \mathbf{x}$ . Moreover we parametrize the unit vector  $\hat{\mathbf{u}} \in \mathcal{C}_0$  as follows

$$\hat{\mathbf{u}} = \left( \mu, \nu, \sqrt{1 - \mu^2 - \nu^2} \right), \quad (\mu, \nu) \in \mathcal{D}_0 \equiv \{(a, b) \in \mathbb{R}^2, a^2 + b^2 < \sin^2 \theta_0\} \tag{8.22}$$

Therefore the integral (8.19) is rewritten as

$$\begin{aligned} & \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{\mathcal{C}_0} d\hat{\mathbf{u}} L_{\underline{n}, \underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^k(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x})} = \\ & \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{\mathcal{D}_0} d\mu d\nu \tilde{L}_{\underline{n}, \underline{m}}(\xi, \eta, s, \sigma, \mu, \nu; \mathbf{x}) e^{\frac{i}{\varepsilon} \tilde{\Theta}_{\underline{n}, \underline{m}}^k(\xi, \eta, s, \sigma, \mu, \nu; \mathbf{x})} \end{aligned} \quad (8.23)$$

where

$$\tilde{L}_{\underline{n}, \underline{m}}(\xi, \eta, s, \sigma, \mu, \nu; \mathbf{x}) = \frac{1}{\sqrt{1 - \mu^2 - \nu^2}} L_{\underline{n}, \underline{m}}(\mathcal{R}_k^{-1} \xi, \mathcal{R}_k^{-1} \eta, s, \sigma; \mathbf{x}) \quad (8.24)$$

$$\begin{aligned} \tilde{\Theta}_{\underline{n}, \underline{m}}^k(\xi, \eta, s, \sigma, \mu, \nu; \mathbf{x}) = & -(\xi_3 + \eta_3)|\mathbf{a}_k| + v_0 \mu(x_1^k + \sigma \xi_1 + s \eta_1) \\ & + v_0 \nu(x_2^k + \sigma \xi_2 + s \eta_2) + v_0 \sqrt{1 - \mu^2 - \nu^2}(x_3^k + \sigma \xi_3 + s \eta_3) + (|n| - |m|)s + |m|\sigma \end{aligned} \quad (8.25)$$

Let us introduce the following linear change of coordinates

$$(\mu, \nu, s) = L_\varepsilon^1(z_1, z_2, z_3) \equiv L_\varepsilon^1 \mathbf{z} \quad \sigma = L_\varepsilon^2 p \quad (8.26)$$

$$\mu = \frac{\varepsilon}{v_0 \tau_k} z_1, \quad \nu = \frac{\varepsilon}{v_0 \tau_k} z_2, \quad s = \tau_k + \frac{\varepsilon}{v_0} z_3 \quad \sigma = \tau_k + \frac{\varepsilon}{v_0} p \quad (8.27)$$

The domain of integration in the variables  $\mathbf{z}, p$  is

$$\Lambda_\varepsilon = \{ \mathbf{z} \in \mathbb{R}^3 \mid p \in \mathbb{R} \mid z_1^2 + z_2^2 < \varepsilon^{-2} v_0^2 \tau_k^2 \sin^2 \theta_0, \quad -\varepsilon^{-1} v_0 \tau_k < z_3, p < \varepsilon^{-1} v_0 (t - \tau_k) \} \quad (8.28)$$

Using (8.26), (8.23) in (8.18) we obtain

$$\begin{aligned} \|J_k^\varepsilon(t) \Psi_{0,k}^\varepsilon\|^2 &= \frac{\mathcal{N}_\varepsilon^2}{\varepsilon^2} \sum_{\underline{n}} \left| \sum_{\underline{m}} \int_0^t ds \int_0^s d\sigma \int d\xi \int d\eta \int_{\mathcal{C}_k} d\hat{\mathbf{u}} L_{\underline{n}, \underline{m}}(\xi, \eta, s, \sigma; \mathbf{x}) e^{\frac{i}{\varepsilon} \Theta_{\underline{n}, \underline{m}}^k(\xi, \eta, s, \sigma, \hat{\mathbf{u}}; \mathbf{x})} \right|^2 \\ &= \frac{\varepsilon^6 \mathcal{N}_\varepsilon^2}{v_0^8 \tau_k^4} \sum_{\underline{n}} \sum_{\underline{m}, \underline{m}'} \int_{\Lambda_\varepsilon} d\mathbf{z} dp \int d\xi \int d\eta \int_{\Lambda_\varepsilon} d\mathbf{z}' dp' \int d\xi' \int d\eta' \tilde{L}_{\underline{n}, \underline{m}}(\xi, \eta, L_\varepsilon^1 \mathbf{z}, L_\varepsilon^2 p; \mathbf{x}) \\ &\quad \cdot \tilde{\tilde{L}}_{\underline{n}, \underline{m}'}(\xi', \eta', L_\varepsilon^1 \mathbf{z}', L_\varepsilon^2 p'; \mathbf{x}) e^{i\mathbf{z} \cdot \boldsymbol{\eta} + ip\xi_3} e^{iz_1 \left( \frac{x_1^k}{\tau_k} + \xi_1 \right) + iz_2 \left( \frac{x_2^k}{\tau_k} + \xi_2 \right) + iz_3 \frac{|n| - |m|}{v_0} + ip \frac{|m|}{v_0} + iA^\varepsilon(\mathbf{z}, p, \xi_3, \eta_3; \mathbf{x})} \\ &\quad \cdot e^{-i\mathbf{z}' \cdot \boldsymbol{\eta}' - ip'\xi_3'} e^{-iz_1' \left( \frac{x_1^k}{\tau_k} + \xi_1' \right) - iz_2' \left( \frac{x_2^k}{\tau_k} + \xi_2' \right) - iz_3' \frac{|n| - |m|'}{v_0} - ip' \frac{|m|'}{v_0} - iB^\varepsilon(\mathbf{z}', p', \xi_3', \eta_3'; \mathbf{x})} \end{aligned} \quad (8.29)$$

where

$$B^\varepsilon(\mathbf{z}, p, \xi_3, \eta_3; \mathbf{x}) = \frac{\sqrt{1 - \left( \frac{\varepsilon z_1}{v_0 \tau_k} \right)^2 - \left( \frac{\varepsilon z_2}{v_0 \tau_k} \right)^2} - 1}{\varepsilon^2} (v_0 x_3^k + |\mathbf{a}_k| \xi_3 + |\mathbf{a}_k| \eta_3 + \varepsilon z_3 \eta_3 + \varepsilon p \xi_3) \quad (8.30)$$

Now we compute the sum over  $\underline{n}, \underline{m}, \underline{m}'$  exploiting (8.12), (8.13) and we obtain

$$\|J_k^\varepsilon(t) \Psi_{0,k}^\varepsilon\|^2 \equiv \frac{\varepsilon^6 \mathcal{N}_\varepsilon^2}{(2\pi)^6 v_0^8 \tau_k^4} \int_{\Lambda_\varepsilon} d\mathbf{z} dp \int_{\Lambda_\varepsilon} d\mathbf{z}' dp' E_k(\mathbf{z}, p, \mathbf{z}', p') \quad (8.31)$$

where

$$\begin{aligned}
E_k(\mathbf{z}, p, \mathbf{z}', p') &= \int d\xi \int d\eta \int d\xi' \int d\eta' \zeta_{b_1, b_2, b_3}(\mathcal{R}_k^{-1}\xi', \mathcal{R}_k^{-1}\eta', \mathcal{R}_k^{-1}\eta, \mathcal{R}_k^{-1}\xi) \\
&e^{\frac{3i}{2v_0}(p-p')} \widetilde{V}(\mathcal{R}_k^{-1}\xi) \widetilde{V}(\mathcal{R}_k^{-1}\xi') \widetilde{V}(\mathcal{R}_k^{-1}\eta) \widetilde{V}(\mathcal{R}_k^{-1}\eta') e^{i(\eta+\xi)\cdot\mathbf{x}^k} e^{-i(\eta'+\xi')\cdot\mathbf{x}^k} \\
&e^{i\frac{\tau_k}{2}(\eta^2+\xi^2)+i\tau_k\eta\cdot\xi} e^{-i\frac{\tau_k}{2}(\eta'^2+\xi'^2)-i\tau_k\eta'\cdot\xi'} e^{i\frac{\varepsilon}{2v_0}(z_3\eta^2+p\xi^2)+i\frac{\varepsilon z_3 p}{v_0}\eta\cdot\xi} e^{i\frac{\varepsilon}{2v_0}(z'_3\eta'^2+p'\xi'^2)+i\frac{\varepsilon z'_3 p'}{v_0}\eta'\cdot\xi'} \\
&f\left(\mathbf{x}^k + \tau_k\eta + \tau_k\xi + \frac{\varepsilon}{v_0}z_3\eta + \frac{\varepsilon}{v_0}p\xi\right) f\left(\mathbf{x}^k + \tau_k\eta' + \tau_k\xi' + \frac{\varepsilon}{v_0}z'_3\eta' + \frac{\varepsilon}{v_0}p'\xi'\right) \\
&e^{i\mathbf{z}\cdot\eta+ip\xi_3} e^{iz_1\left(\frac{x_1^k}{\tau_k}+\xi_1\right)+iz_2\left(\frac{x_2^k}{\tau_k}+\xi_2\right)+iB^\varepsilon(\mathbf{z}, p, \xi_3, \eta_3; \mathbf{x})} \\
&e^{-i\mathbf{z}'\cdot\eta-ip'\xi'_3} e^{-iz'_1\left(\frac{x_1^k}{\tau_k}+\xi'_1\right)-iz'_2\left(\frac{x_2^k}{\tau_k}+\xi'_2\right)-iB^\varepsilon(\mathbf{z}', p', \xi'_3, \eta'_3; \mathbf{x})}
\end{aligned} \tag{8.32}$$

and  $b_1 = (p' - z'_3)/v_0$ ,  $b_2 = (z'_3 - z_3)/v_0$ ,  $b_3 = (z_3 - p)/v_0$ .

It remains to show that the integrals in (8.35) are bounded. Exploiting the identity

$$\partial_{\eta_l}^4 \partial_{\xi_3}^4 [e^{i\mathbf{z}\cdot\eta+ip\xi_3}] \partial_{\eta'_l}^4 \partial_{\xi'_3}^4 [e^{-i\mathbf{z}'\cdot\eta-ip'\xi'_3}] = z_l^4 p^4 z'_l{}^4 p'^4 [e^{i\mathbf{z}\cdot\eta+ip\xi_3}] [e^{-i\mathbf{z}'\cdot\eta-ip'\xi'_3}] \quad l = 1, 2, 3 \tag{8.33}$$

we can integrate by parts and we have

$$\langle \mathbf{z} \rangle^4 \langle \mathbf{z}' \rangle^4 \langle p \rangle^4 \langle p' \rangle^4 |E_k(\mathbf{z}, p, \mathbf{z}', p')| \leq c \|\widetilde{V}\|_{W_4^{4,1}}^4 \tag{8.34}$$

Finally we use the estimate (8.34) in (8.35) and we find

$$\begin{aligned}
\|J_k^\varepsilon(t) \Psi_{0,k}^\varepsilon\|^2 &\leq c \varepsilon^6 \int_{\Lambda_\varepsilon} d\mathbf{z} dp \int_{\Lambda_\varepsilon} d\mathbf{z}' dp' E_k(\mathbf{z}, p, \mathbf{z}', p') \\
&\leq c \varepsilon^6 \|\widetilde{V}\|_{W_4^{4,1}}^4 \int d\mathbf{z} d\mathbf{z}' \frac{1}{\langle \mathbf{z} \rangle^4 \langle \mathbf{z}' \rangle^4} \int dp dp' \frac{1}{\langle p \rangle^4 \langle p' \rangle^4} \leq c \varepsilon^6 \|\widetilde{V}\|_{W_4^{4,1}}^4
\end{aligned} \tag{8.35}$$

concluding the proof of the proposition.

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